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Modelling Miraa Addiction like a Disease Incorporating Voluntary Quitting

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Authors' contributions

This work was carried out in collaboration between all authors. Author MG designed the study, carried out literature searches, wrote the algorithm, and wrote the first draft of the manuscript. Authors KS and DT guided in the analyses of the study and interpretation of the results. All authors read and approved the final manuscript.

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Abstract

This study presented a deterministic model of miraa addiction based on three compartmental classes incorporating miraa specific attributes as well as the aspect of voluntary quitting. Our model was based on SIS classical infectious model classes with Susceptible(S) and Infected (I) adopted as Light user (L) and Addicted (A) in our model. From the model flow chart, non linear differential equations are deduced. The basic reproduction number (R_0) was determined using next generation method. Positivity and boundedness of the solution was investigated and the system of equations was found to lie in the feasible region. Miraa equilibrium points were determined and the condition necessary for the existence of miraa persistent equilibrium point was found to be $R_0 > 1$. The conditions necessary for both local and global asymptotic stability of equilibrium points were determined. Sensitivity analysis of the R_0 was investigated using partial differentiation and then confirmed using normalized sensitivity analysis. Simulations were carried out using MATLAB ODE 45 inbuilt solver. Sensitivity analysis results revealed that the R_0 was directly proportional to the rate of quitting from addict to light user but inversely

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proportional to the rate of quitting from light user to susceptible. Therefore the rate of individuals moving from light user class to susceptible classes has higher impact on reducing the burden of miraa addiction than the rate of individuals moving from addict to light user. This study used theoretical data and parameters, future studies should consider fitting model to real data. The findings of this study will provide the stake holders including the government, NACADA, rehabilitation centres and general public with information of the spread of the addiction so that necessary measures may be taken to address the challenges. The model can find application in predicting future trends which is necessary for planning. Control strategies can be instituted with the help of the model.

Keywords: Modelling; basic reproduction number; stability; normalized sensitivity analysis and simulation.

1 Background

Miraa (Catha edulis) is a plant whose twigs and leaves are chewed to release a juice containing cathinone and cathine which are active chemicals that alter the mood of the user. In Africa, it originated from Ethiopia and spread to Somali, Djiouti, Kenya, Uganda, Tanzania, Congo, Malawi, Zimbambwe, Zambia and South Africa. In Kenya, it's referred by different names depending on the place. These includes: khat, veve, muguka, guks, gomba, mbachu, mairungi, alele, giza or halwa. Miraa is a source of livelihood for very many Kenyan citizens either through provision of security in the farms, production, harvesting, processing, transportation or packaging for marketing. It also provides indirect employment to many people who are involved. Originally, it was used by specific group of people in special occasions like marriage ceremonies and appeasing communities [1].

According to the rapid situation assessment of status of drug and substance abuse in Kenya, today miraa is readily available and people use it for various reasons ranging from peer pressure, need to stay awake and increase energy to work for long hours, suppress fatigue, brighten up mood, idleness or because it is cheap compared to other alternatives [2].

Despite the benefits of miraa, the studies carried out have revealed that the habit of chewing miraa damages health and affects many aspects of life with adverse social, economic and medical consequences. The study by Hassan et al. [3] on the health aspects of khat chewing examined the effects of khat on the body systems and its relationship with common diseases. It concluded that chewing khat poses a number of health risks which includes stimulating the central nervous system bringing about mood disturbances. It causes lack of sleep or increased alertness, anxiety, depression and insomnia. This affects individual's general physical and mental well-being given that this kind of alertness is induced. Habitual miraa chewing also affects the digestive system. Anorexia (loss of appetite) often follows a miraa chewing session. Constipation is also a common complaint among users [4,5].

Chewing miraa is also associated with tooth decay which is worse among the people who use it with sugary accompaniments like soda, sweetened coffee or tea. Long-term miraa chewing can cause an inflammation of the parts of mucous membrane lining of the mouth (stomatitis) followed by infections. High rate of periodontal diseases and low rate of dental caries has been observed among miraa chewers [6].

Use of miraa has certain social economic implications to the users, their households and community at large. It demands availability of financial resources and since it's addictive, users have to make difficult choices to satisfy the craving. It is further associated with wastage of productive time (due to idling), increased possibility of using other drugs and substances of abuse. A key concern by policy makers in the area of drugs and substance abuse has been the linkage between miraa and use of other drugs and substance of abuse. Taking various forms of alcohol and smoking often accompany miraa chewing [7].

From the foregoing, some regulation is required on production, sale and consumption of miraa in order to confer benefits to the producer and ensure safety to the consumer. It is upon this background that a deterministic model is required to understand the dynamics of miraa addiction in order to empower individuals on negative effects of khat to help them make informed decisions.

The field of drug addiction has been of interest to many researchers, since it is a major problem in the society. Various studies have been carried out about miraa addiction. The following are some studies done on miraa addiction.

A research by Madrine [8] studied the extent of substance abuse in Kenya. Statistical random sampling techniques were used to obtain data and analyzed using statistical package for social science. The study found out that drug abuse is a big threat to the Kenyan society. Crime rates were directly related to drug abuse. Various behavioral disorders were witnessed among the youth besides beastly acts such as rape and murder of innocent people.

A study by Nelly et al. [9] investigated the effects of khat production on rural household's income. Probability and non- probability sampling techniques were used to collect and analyze the information. The factors that influence diversification into khat production and its contribution to rural households in Kenya were examined. Khat production was found to positively contribute to the people's incomes.

Another research by Jane et al. [10] investigated the participation of women in miraa business in relation to the academic performance of children. This study concluded that the participation in khat business impacted negatively on the academic performance of the children whose parents are involved. A study on the effect of occasional smokers on the dynamics of smoking model analyzed a generalized giving up smoking model and found out that quitting smoking can be temporary or permanent. Their population with peer pressure influence on temporary quitters was analyzed to check the possibility of such becoming permanent quitters and the impact of this transformation on the existence and stability of equilibrium points [11].

Addiction to miraa is currently a major problem in our society. It has led to the reduction in productivity, loss of life and health complications. .Mathematical modeling is a vital tool that could be used to provide an understanding of patterns and underlying mechanisms which influence the spread of the habit [12].

There has not been a developed deterministic mathematical model to give insight on the dynamics of miraa addiction. This research study uses mathematical modeling approach where the population is subdivided into three compartmental classes. The rate of change from one class to another is formulated in nonlinear ordinary differential equations and expression for reproduction number determined. Model analysis is carried out and and assumed parameters are fitted in the developed model using MATLAB software. The estimated numerical results such as the reproduction number and numerical sensitivity analysis are obtained to validate the model. Numerical simulation to predict the dynamics of miraa consumption is carried out using theoretical data and parameters.

2 Model Formulation

A classical endemic initial value problem of SIS includes three classes from which the equations below can be deduced [13].

$$\dot{S} = \pi + \rho I - \frac{\beta S I}{N} - \mu S$$

$$\dot{I} = \frac{\beta S I}{N} - (\mu + \delta + \rho) I$$

where, μ is constant death rate, δ is disease induced death rate, ρ is the recovery rate, $\frac{\beta S I}{N}$ is the force of infection, S(t) and I(t) are the numbers in the susceptible and infected compartments respectively, so that $S(t) + I(t) = N$.

From the above classical model, the infected, I compartmental class is divided into two classes to get a general deterministic model for miraa addiction with three compartmental classes, which are Susceptible, Light users and Addicted. From the flow chart, three nonlinear first order ordinary differential equations that governed the dynamics of miraa addiction are deduced.

A study on drug dependence, its significance and characteristics assessed khat and found that it causes moderate but often persistent psychic dependence [14]. In our model, the individuals with moderate dependence are classified in a compartment of light users (L).

A qualitative study in Saudi concerning former khat users analyzed the initiation, continuance and cessation. It was found out that there are multiple reasons for quitting. These included feeling lost and neglecting family, accumulating debts, feeling guilty, absenteeism and neglect of work, health effects among others [15,16]. In our model, this voluntary quitting was incorporated by considering moving from addict class to light users as well as light user to susceptible.

From the SIS model, the infected compartmental class, I is divided into two classes to get a deterministic model for miraa addiction with three compartmental classes, which are Susceptible, Light users and Addicted. Quitting from light user to susceptible and from addict to light user group is incorporated.

Let N(t) be the total population which is sub-divided into three sub-groups depending on the attributes for miraa addiction as:

S(t) = Number of susceptible persons
L(t) = Number of light users
A(t) = Number of Addicts

The rate at which susceptible recruit to light users is given by λS while the rate at which the light users become addict is given by τL. The rate of quitting from addict to light user is given by αA while from light users to susceptible is θL. The exit rate and natural death rate is given by γ and μ respectively while miraa induced death rate is δ.

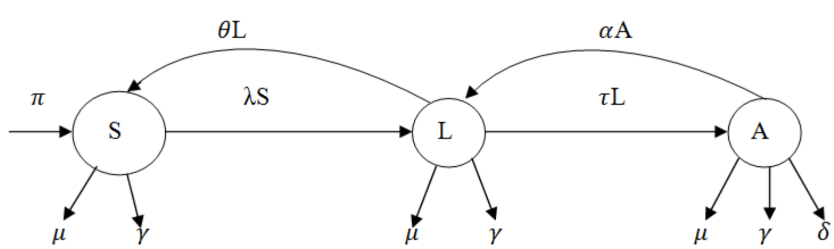
The following assumptions are made:

i). There is homogenous mixing of the population
ii). The effect of immigration and emigration are assumed not significant
iii). The light users have more chances of recruiting others

$$\lambda(t) = \beta(L + \eta A) \quad , \eta < 0.5$$

Where λ = force of recruitment
β = recruitment rate

The flow chart diagram is as follows:



From the flow chart above, we obtain the following system of equations representing the population dynamics of miraa addiction;

$$\frac{dS}{dt} = \pi + \theta L - \lambda S - \omega_1 S \quad (1)$$

$$\frac{dL}{dt} = \lambda S + \propto A - \omega_2 L \quad (2)$$

$$\frac{dA}{dt} = \tau L - \omega_3 A \quad (3)$$

Where, $N(t) = S(t) + L(t) + A(t)$

$$\omega_1 = (\mu + \gamma)$$
$$\omega_2 = (\tau + \mu + \theta + \gamma)$$
$$\omega_3 = (\propto + \mu + \gamma + \delta)$$

The starting conditions of the systems were represented by;

$$S(0) = S_0, L(0) = L_0 \text{ and } A(0) = A_0.$$

The force of recruitment denoted by $\lambda(t)$ is given by

$$\lambda(t) = \beta(L + \eta A), \text{where}, \quad \eta < 0.5.$$

The sum of equations of the system $[(1) - (3)]$, the rate of change of total population is given by

$$\frac{dN}{dt} = \pi - (\mu + \gamma)N - \delta A.$$

3 Model Analysis

The model is analyzed by proving various theorems and carrying out algebraic computations dealing with different attributes.

3.1 Positivity and boundedness of the solutions

Theorem 1: The region given by $R = \left\{S(t), L(t), A(t) \in R_+^3 ; N(t) \leq \frac{\pi}{\omega_1}\right\}$ is positively invariant and attracting with respect to model system $[(1) - (3)]$.

Proof.

Let $\{S(t), L(t) \text{ and } A(t)\}$ be any solutions of the system with non-negative initial conditions $\{S(0) \geq 0, L(0) \geq 0, A(0) \geq 0\}$.

Since

$$\frac{dS}{dt} = \pi + \theta L - \lambda S - \omega_1 S$$

it follows that $\frac{dS}{dt} \geq -(\lambda + \omega_1)S$.

On integration, we obtain

$$\frac{d}{dt}[S(t)e^{\int_o^t -(\lambda+\omega_1)ds}] \geq 0.$$

Clearly, $S(t)e^{\int_o^t -(\lambda+\omega_1)ds}$is a non-negative function of t, thus S (t) stays positive.

The positivity of $L(t)$ and $A(t)$ is proved along the same lines as follows:

$$\frac{dL}{dt} = \lambda S + \propto A - \omega_2 L$$

$$\frac{dL}{dt} \geq -\omega_2 L$$

$$L(t) \geq L(0)e^{-\omega_2 t} \geq 0.$$

Similarly, $\frac{dA}{dt} = \tau L - \omega_3 A$

$$\frac{dA}{dt} \geq -\omega_3 A$$

$$A(t) \geq A(0)e^{-\omega_3 t} \geq 0.$$

Taking the time derivative of our total population along its solution path gives:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dL}{dt} + \frac{dA}{dt}$$

$$\frac{dN}{dt} = \pi - \omega_1 N - \delta A$$

Therefore,

$$\frac{dN}{dt} + \omega_1 N \leq \pi$$

This implies that

$$N(t) \leq \left\{\frac{\pi}{\mu + \gamma} + C_3 e^{-\mu t}\right\}$$

Where C_3 is a constant of integration.

Hence

$$\lim_{t\to\infty} N(t) \leq \frac{\pi}{\omega_1}$$

This proves the boundedness of the solutions inside R. This implies that all the solutions of our system$[(1) - (3)]$, starting in R and will remains in R for all $t \geq 0$. Thus R is positively invariant and attracting, and hence it is sufficient to consider the dynamics of our system in R. This completes the proof.

3.2 Miraa-free equilibrium point (MFE)

The Miraa Free Equilibrium point (MFE) of the system$[(1)-(3)]$ is obtained by setting the light users and addicts classes to zero.

This gives $\pi-\omega_1 S=0$

which yield $S^0=\frac{\pi}{\omega_1}$.

The MFE point for our system is given by

$$E^0=(S^0,L^0,A^0)=\left(\frac{\pi}{\omega_1},0,0\right).$$

The point E^0 is the miraa free equilibrium point (MFE) of the system$[(1)-(3)]$, which indicates that in absence of miraa use, the system$[(1)-(3)]$ will consist of only one compartment class (susceptible).

3.3 The basic reproduction number (R_o)

The basic reproduction number of the system was determined using the next generation method, whereby the Jacobi matrix of the system of equations was evaluated at MFE and the spectral radius determined. There exists various methods of determining the basic reproduction number (R_o).These includes; use of survival function, use of the Jacobian, use of constant term of the characteristic polynomial, using next generation matrix, graph-theoretic method among others.

We used the next-generation matrix method to determine the basic reproduction number (R_o) of the model [17,18,19,20]. This is because the method is significantly easier as it requires only the infection states and ignores the other states such as susceptible and recovered. This keeps the size of the matrices relatively manageable.

Using the notation f for a matrix of new recruits terms and v for the matrix of the remaining transfer of recruits terms in our system, we get,

$$f=\begin{pmatrix}\lambda S\\0\end{pmatrix},\quad v=\begin{pmatrix}-\propto A+\omega_2 L\\-\tau L+\omega_3 A\end{pmatrix}.$$

We let

$$F_1=\lambda S,\qquad F_2=0,\qquad F_3=-\propto A+\omega_2 L \text{ and } F_4=-\tau L+\omega_3 A,$$

We obtain the matrices F and V by finding the Jacobian matrices of *f* and v evaluated at MFE respectively to obtain,

$$F=\begin{pmatrix}\frac{d}{dL}F_1 & \frac{d}{dA}F_1\\ \frac{d}{dL}F_2 & \frac{d}{dA}F_2\end{pmatrix}=\begin{pmatrix}\beta S^0 & \beta n S^0\\ 0 & 0\end{pmatrix}$$

and

$$V=\begin{pmatrix}\frac{d}{dL}F_3 & \frac{d}{dA}F_3\\ \frac{d}{dL}F_4 & \frac{d}{dA}F_4\end{pmatrix}=\begin{pmatrix}\omega_2 & -\propto\\ -\tau & \omega_3\end{pmatrix}$$

Computing the inverse of V we obtain,

$$V^{-1} = \frac{1}{\omega_2\omega_3 - \propto \tau}\begin{pmatrix} \omega_3 & \propto \\ \tau & \omega_2 \end{pmatrix}.$$

Multiplying the matrices F and V^{-1} we obtain,

$$FV^{-1} = \frac{1}{\omega_2\omega_3 - \propto \tau}\begin{pmatrix} \beta S^0\omega_3 + \beta n S^0\tau & \beta S^0 \propto + \beta n S^0\omega_2 \\ 0 & 0 \end{pmatrix}$$

We get the Eigen values as $\frac{\beta S^0\omega_3 + \beta n S^0\tau}{\omega_2\omega_3 - \propto\tau}$ and 0.

The basic reproduction number (R_0) is given by the spectral radius ζ (the dominant eigenvalue) of the matrix FV^{-1}, denoted by $\zeta(FV^{-1})$ which is

$$R_0 = \frac{(\omega_3 + n\tau)\beta S^0}{\omega_2\omega_3 - \propto \tau}.$$

The basic reproduction number(R_0) is the average number of susceptible persons one miraa consumer (light user or addict) can recruit in a susceptible population.

3.4 Existence of Miraa persistent equilibrium point for the model (MPE)

The condition necessary for the force of recruitment to be positive was evaluated to predict the presence of the miraa persistent equilibrium point.

Theorem 2: A positive Miraa Persistent Equilibrium point exists and is locally asymptotically stable whenever $R_0 > 1$.

Proof

The dynamic systems of the equations$[(1) - (3)]$ are three dimensional.

Let the miraa persistent equilibrium point be $(E^* = (S^*, L^*, A^*)$ of the system $[(1) - (3)]$and the force of recruitment be given by $\lambda^* = \beta(L^* + \eta A^*)$.

Upon solving the system of equation above $[(1) - (3)]$ in terms of λ^* using Mathematica software we obtain:

$$S^* = \frac{\pi(\alpha\tau - \omega_2\omega_3)}{\alpha\tau(\lambda + \omega_1) + (\theta\lambda - (\lambda + \omega_1)\omega_2)\omega_3}$$

$$L^* = -\frac{\pi\lambda\omega_3}{\alpha\tau(\lambda + \omega_1) + (\theta\lambda - (\lambda + \omega_1)\omega_2)\omega_3}$$

$$A^* = -\frac{\pi\lambda\tau}{\alpha\pi\lambda + \alpha\tau\omega_1 + \theta\lambda\omega_3 - \lambda\omega_2\omega_3 - \omega_1\omega_2\omega_3}$$

Substituting $A^*$and L^* using Mathematica software to solve the equation below we obtain two cases:

$$\lambda^* = -\frac{\pi\beta\lambda^*(\eta\tau + \omega_3)}{\alpha\tau(\lambda^* + \omega_1) + (\theta\lambda^* - (\lambda^* + \omega_1)\omega_2)\omega_3}$$

Case 1. $\lambda^* = 0$, which corresponds to the miraa free equilibrium point(E^0) of the system $[(1) - (3)]$ which was given by

$$E^0 = (S^0, L^0, A^0) = \left(\frac{\pi}{\omega_1}, 0, 0\right).$$

Case 2. The value(s) of λ^* obtained by the linear equations below correspond to miraa persistent equilibrium point, which is solved as

$$\lambda^* = \frac{-\pi\beta\eta\tau - \alpha\tau\omega_1 - \pi\beta\omega_3 + \omega_1\omega_2\omega_3}{\alpha\tau + \theta\omega_3 - \omega_2\omega_3}$$

$$\lambda^* = \frac{(\pi R_0 - S^0\omega_1)(\alpha\tau - \omega_2\omega_3)}{S^0\alpha\tau + S^0(\theta - \omega_2)\omega_3}$$

Where

$$\omega_1 = (\mu + \gamma)$$

$$\omega_2 = \tau + \mu + \theta + \gamma$$

$$\omega_3 = (\propto + \mu + \gamma + \delta)$$

$$\lambda^* = \frac{(\pi R_0 - S^0\omega_1)(\alpha\tau - \omega_2\omega_3)}{S^0\alpha\tau + S^0(\theta - \omega_2)\omega_3} > 0$$

After algebraic manipulation it follows that the conditions necessary and sufficient for $\lambda^* > 0$ is $R_0 > 1$ which completes the proof.

The expression for miraa persistent equilibrium point is evaluated by substituting the expression for λ^*

$$S^* = \frac{S^0}{R_0}$$

$$L^* = \frac{(\pi R_0 - S^0\omega_1)\omega_3}{R_0(\alpha\tau + (\theta + \omega_2)\omega_3)}$$

$$A^* = \frac{-\pi\tau R_0 + S^0\tau\omega_1}{R_0(\alpha\tau + (\theta + \omega_2)\omega_3)}$$

3.5 Local stability of the miraa free equilibrium point (MFE)

The signs of Eigen values of the Jacobi matrix evaluated at MFE were used to determine the condition necessary for the local asymptotically stability of the miraa free equilibrium point.

Theorem 3: The MFE of the system $[(1) - (3)]$is locally asymptotically stable whenever $R_0 < 1$ and unstable otherwise.

Proof

To establish the local stability of the system$[(1) - (3)]$, we use the Jacobian of the model evaluated at E^0. Stability of this steady state can then be determined based on the signs eigenvalues of the corresponding Jacobian which are functions of the model parameters.

We let

$$P_1 = \pi + \theta L - \lambda S - \omega_1 S$$

$$P_2 = \lambda S + \propto A - \omega_2 L,$$

$$P_3 = \tau L - \omega_3 A\ ,$$

$$\lambda(t) = \beta(L + \eta A)$$

The Jacobian matrix evaluated at miraa free equilibrium point E^0 is obtained as

$$J(E^0) = \begin{pmatrix} -\omega_1 & \theta - \beta S^0 & -\eta\beta S^0 \\ 0 & \beta S^0 - \omega_2 & \eta\beta S^0 + \propto \\ 0 & \tau & -\omega_3 \end{pmatrix}$$

The eigenvalues, $i = (1) - (3)$ of the above matrix using Mathematica software are obtained as follows:

$$q(1) = -\omega_1$$

$$q(2) = \frac{1}{2}\left(-\omega_2 - \omega_3 + \frac{R_0(-\alpha\tau + \omega_2\omega_3)}{\eta\tau + \omega_3} - \sqrt{\left(-4\langle -1 + R_0\rangle\langle \alpha\tau - \omega_2\omega_3\rangle + \left(\omega_2 + \omega_3 + \frac{R_0(\alpha\tau - \omega_2\omega_3)}{\eta\tau + \omega_3}\right)^2\right)}\right)$$

$$q(3) = \frac{1}{2}\left(-\omega_2 - \omega_3 + \frac{R_0(-\alpha\tau + \omega_2\omega_3)}{\eta\tau + \omega_3} + \sqrt{\left(-4\langle -1 + R_0\rangle\langle \alpha\tau - \omega_2\omega_3\rangle + \left(\omega_2 + \omega_3 + \frac{R_0(\alpha\tau - \omega_2\omega_3)}{\eta\tau + \omega_3}\right)^2\right)}\right)$$

Clearly the first eigenvalue is negative but the condition necessary and sufficient for $q(2)$ and $q\ (3)$ to be negative is $R_0 < 1$. This is determined using Mathematica software algebraic computation.

This completed the proof.

3.6 Global stability of the miraa free point (MFE)

A Lyapunov criterion for stability is used to determine the condition necessary for the global asymptotic stability of the miraa free equilibrium point.

Theorem 4: The MFE is globally asymptotically stable in Lyapunov sense whenever $R_0 < 1$ and unstable otherwise.

Proof

We propose the following Lyapunov function for the system $[(1) - (3)]$

$$K(S, L, A) = S - S^0 - S^0 \mathrm{Ln}\frac{S}{S^0} + X_1 L + X_2 A$$

where X_1 and X_2 were positive constants to be determined at MFE point.

$K(S, L, A)$ is positive definite and satisfies the conditions

$$K(S^0, L^0, A^0) = 0 \text{ and } K(S, L, A,) > 0.$$

For $\frac{dK(S,L,A,)}{dt}$ to be negative definite, it must satisfy

$$\frac{dK(S^0, L^0, A^0)}{dt} = 0 \text{ and } \frac{dK(S, L, A)}{dt} < 0.$$

$$E^0 = (S^0, L^0, A^0,)$$

The system $[(1) - (3)]$ satisfy, $\pi = \omega_1 S^0$.

The time derivative of the Lyapunov function is obtained as

$$\frac{dK(S, L, A)}{dt} = \left(1 - \frac{S^0}{S}\right)\frac{dS}{dt} + X_1\frac{dL}{dt} + X_2\frac{dA}{dt},$$

Substituting $\frac{dS}{dt}, \frac{dL}{dt}$ and $\frac{dA}{dt}$ to obtain;

$$\frac{dK(S, L, A)}{dt} = \left(1 - \frac{S^0}{S}\right)(\pi + \theta L - \lambda S - \omega_1 S) + X_1(\lambda S + \propto A - \omega_2 L) + X_2(\tau L - \omega_3 A),$$

Substituting π to obtain;

$$\frac{dK(S, L, A)}{dt} = \left(1 - \frac{S^0}{S}\right)(\omega_1 S^0 + \theta L - \lambda S - \omega_1 S) + X_1(\lambda S + \propto A - \omega_2 L) + X_2(\tau L - \omega_3 A),$$

$$\frac{dL(S, V, A, R)}{dt} = -\omega_1\frac{(S - S^0)^2}{S} - \theta L\frac{S^0}{S} + \lambda S + \theta L - \lambda S^0 + X_1(\lambda S + \propto A - \omega_2 L) + X_2(\tau L - \omega_3 A),$$

$$\frac{dL(S, V, A, R)}{dt} = -\omega_1\frac{(S - S^0)^2}{S} - \theta L\frac{S^0}{S} + \{1 + X_1\}\lambda S + \{\theta - \beta S^0 - X_1\omega_2 + \tau X_2\}L + \{\propto X_1 - \beta\eta S^0 - X_2\omega_3\}A$$

Setting λS and L to zero we obtain the following equation,

$$1 + X_1 = 0, \qquad X_1 = -1,$$
$$\theta - \beta S^0 + \omega_2 + \tau X_2 = 0, \quad X_2 = \frac{\beta S^0 - \omega_2 - \theta}{\tau}.$$

The derivative of the Lyapunov reduces to;

$$\frac{dK(S, L, A)}{dt} = -\omega_1\frac{(S - S^0)^2}{S} - \theta L\frac{S^0}{S} + \left\{-\propto - \beta\eta S^0 - \frac{\beta S^0\omega_3 - \omega_2\omega_3 - \theta\omega_3}{\tau}\right\}A$$

After introducing R_O in the above equation we obtain

$$\frac{dK(S, L, A)}{dt} = -\omega_1\frac{(S - S^0)^2}{S} - \theta L\frac{S^0}{S} + \left\{-\propto - \beta\eta S^0 - \frac{\beta S^0\omega_3 - \omega_2\omega_3 - \theta\omega_3}{\tau}\right\}A$$

$$\frac{dK(S,L,A)}{dt} = -\omega_1 \frac{(S-S^0)^2}{S} - \theta L \frac{S^0}{S} + \left\{\frac{R_O - 1}{\tau(\propto \tau - \omega_2\omega_3)} + \frac{\theta\omega_3}{\tau}\right\} A$$

For $\frac{dK(S,L,A)}{dt} < 0$ then the necessary and sufficient condition is $\frac{R_O - 1}{\tau(\propto\tau - \omega_2\omega_3)} + \frac{\theta\omega_3}{\tau} < 0$ that is

$R_O < 1 - \theta\omega_3(\propto \tau - \omega_2\omega_3)$. By inspection $-\theta\omega_3(\propto \tau - \omega_2\omega_3 \geq 0$ hence the $R_O < 1$ is also a sufficient condition.

This completes the proof.

3.7 Stability of the miraa persistent equilibrium point (MPE)

To determine the asymptotic stability of the miraa persistent Equilibrium point (MPE), we state and prove the following theorem.

Theorem 5: The miraa persistent Equilibrium point (MPE) is stable wheneverP $> Q$, where

$$P = \omega_1 \frac{(s-S^*)^2}{s} + \lambda^* S^* \frac{S^*}{S} + \beta S \frac{L^*}{L}(L + \eta A) + \propto A \frac{L^*}{L} + \tau L \frac{A^*(\beta\eta S^* + \propto)}{A\omega_3} + \omega_2 L + \frac{\omega_3 A(\beta\eta S^* + \propto)}{\omega_3} \text{ and}$$
$$Q = \frac{\omega_3 A^*(\beta\eta S^* + \propto)}{\omega_3} + \beta L S^* + \lambda^* S^* + \omega_2 L^* + \frac{\tau L(\beta\eta S^* + \propto)}{\omega_3}$$

and unstable otherwise.

Proof

We propose the following Lyapunov function

$$H(S, L, A,) = S - S^* - S^* \text{Ln} \frac{S}{S^*} + Y_1 \left(L - L^* - L^* \text{Ln} \frac{L}{L^*}\right) + Y_2 \left(A - A^* - A^* \text{Ln} \frac{A}{A^*}\right)$$

where, Y_1 and Y_2 are positive constants to be determined. The lyapunov function $H(S, L, A)$ satisfies the conditions, $H(S^*, L^*, A^*) = 0$ and $H(S, L, A) > 0$, hence it is positive definite. For $\frac{dH(S,L,A)}{dt}$ to be negative definite, it must satisfy

$$\frac{dH(S^*, L^*, A^*)}{dt} = 0 \quad \text{and} \quad \frac{dH(S, L, A)}{dt} < 0.$$

The Miraa persistent equilibrium point $E^* = (S^*, L^*, A^*)$ for the system satisfies,

$$\pi = -\theta L + \lambda^* S^* + \omega_1 S^* \tag{4}$$

$$\lambda^* S^* + \propto A^* = \omega_2 L^* \tag{5}$$

$$\tau L^* = \omega_3 A^* \tag{6}$$

Determining the time derivative of the Lyapunov equation we obtain,

$$\frac{dH(S,L,A)}{dt} = \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + Y_1\left(1 - \frac{L^*}{L}\right)\frac{dL}{dt} + Y_2\left(1 - \frac{A^*}{A}\right)\frac{dA}{dt}$$

Substituting for $\frac{dS}{dt}, \frac{dL}{dt}$ and $\frac{dA}{dt}$ we get:

$$\frac{dH(S,L,A)}{dt} = \left(1-\frac{S^*}{S}\right)\{\pi+\theta L-\lambda S-\omega_1 S\} + Y_1\left(1-\frac{L^*}{L}\right)\{\lambda S+\propto A-\omega_2 L\} + Y_2\left(1-\frac{A^*}{A}\right)\{\tau L-\omega_3 A\}$$

$$\frac{dH(S,L,A)}{dt} = \left(1-\frac{S^*}{S}\right)\{-\theta L+\lambda^* S^*+\omega_1 S^*+\theta L-\lambda S-\omega_1 S\} + Y_1\left(1-\frac{L^*}{L}\right)\{\lambda S+\propto A-\omega_2 L\} + Y_2\left(1-\frac{A^*}{A}\right)\{\tau L-\omega_3 A\}$$

$$\frac{dH(S,L,A)}{dt} = -\omega_1\frac{(s-S^*)^2}{s} + \left(1-\frac{S^*}{S}\right)\{\lambda^* S^*-\lambda S\} + Y_1\left(1-\frac{L^*}{L}\right)\{\lambda S+\propto A-\omega_2 L\} + Y_2\left(1-\frac{A^*}{A}\right)\{\tau L-\omega_3 A\}$$

Substituting $\lambda = \beta(L+\eta A)$

$$\frac{dH(S,L,A)}{dt} = -\omega_1\frac{(s-S^*)^2}{s} + -\lambda^* S^*\frac{S^*}{S} + \beta(L+\eta A)S^* + Y_1\left\{-\beta S\frac{L^*}{L}(L+\eta A) - \propto A\frac{L^*}{L} + \omega_2 L^*\right\} + Y_2\left\{-\tau L\frac{A^*}{A} + \omega_3 A^*\right\} + \lambda^* S^* - \beta(L+\eta A)S + Y_1\{\beta(L+\eta A)S+\propto A-\omega_2 L\} + Y_2\{\tau L-\omega_3 A\}$$

Setting AS and A to zero, we have

$-\beta\eta AS + Y_1\beta(\eta A)S = 0;\ Y_1 = 1$

$\beta\eta AS^* + Y_1 \propto A - Y_2\omega_3 A = 0; Y_2 = \frac{\beta\eta S^*+\propto}{\omega_3}$

$$\frac{dH(S,L,A)}{dt} = -\left\{\omega_1\frac{(s-S^*)^2}{s} + \lambda^* S^*\frac{S^*}{S} + \beta S\frac{L^*}{L}(L+\eta A) + \propto A\frac{L^*}{L} + \tau L\frac{A^*(\beta\eta S^*+\propto)}{A\omega_3} + \omega_2 L + \frac{\omega_3 A(\beta\eta S^*+\propto)}{\omega_3}\right\} + \left\{\frac{\omega_3 A^*(\beta\eta S^*+\propto)}{\omega_3} + \beta L S^* + \lambda^* S^* + \omega_2 L^* + \frac{\tau L(\beta\eta S^*+\propto)}{\omega_3}\right\}$$

Since $P = \omega_1\frac{(s-S^*)^2}{s} + \lambda^* S^*\frac{S^*}{S} + \beta S\frac{L^*}{L}(L+\eta A) + \propto A\frac{L^*}{L} + \tau L\frac{A^*(\beta\eta S^*+\propto)}{A\omega_3} + \omega_2 L + \frac{\omega_3 A(\beta\eta S^*+\propto)}{\omega_3}$

And $Q = \frac{\omega_3 A^*(\beta\eta S^*+\propto)}{\omega_3} + \beta L S^* + \lambda^* S^* + \omega_2 L^* + \frac{\tau L(\beta\eta S^*+\propto)}{\omega_3}$

This completes the proof.

3.8 Analytical sensitivity analysis of basic reproduction number and their social interpretations

Partial differentiation of the basic reproduction number with respect to θ, the rate of quitting from light users to susceptible and α, the rate of quitting from addicts to light users is carried out. Social interpretations are then provided.

Firstly, differentiating R_o partially with respect to θ we get

$$\frac{\partial R_o}{\partial\theta} = -\frac{\beta\omega_3(\eta\tau+\omega_3)s^0}{(\alpha\tau-(\gamma+\theta+\mu+\tau)\omega_3)^2} < 0$$

This shows that the basic reproduction number is inversely proportional to the rate of quitting from light users to susceptible population. This implies that an increase in the number of people quitting from light users group reduces the rate of recruitment. In order to reduce the burden of miraa consumption therefore more efforts should be made to encourage light users to quit. This will be confirmed later in numerical sensitivity analysis.

Secondly, differentiating R_o partially with respect to α we get

$$\frac{\partial R_o}{\partial \alpha} = \frac{s^0\beta\big(\gamma + \delta - \gamma\eta + \mu - \eta(\theta + \mu)\big)\tau}{\big(\alpha(\gamma + \theta + \mu) + (\gamma + \delta + \mu)(\gamma + \theta + \mu + \tau)\big)^2}$$

The relationship between R_o and α is not tractable from the above expression by observation. However, it will be determined later through numerical sensitivity analysis.

4 Validation of the Model

To validate the model, we assume the initial conditions of the model and the parameters as below Table 1:

Table 1. Summary of the assumed parameters and initial conditions

Parameters	Value	Source	Initial conditions	Value	Source
θ	0.00132	A	$N(0) = S(0) + L(0) + A(0)$	2000	A
α	0.33	A	S(0)	1900	A
γ	0.0476	A	L(0)	990	A
δ	0.00234	A	A(0)	10	A
π	1000	A			
μ	1/63.44	A			
η	0.34	A			
τ	0.0238	A			
β	0.00022	A			

Where, A denotes assumed.

4.1 Numerical value of the basic reproduction number

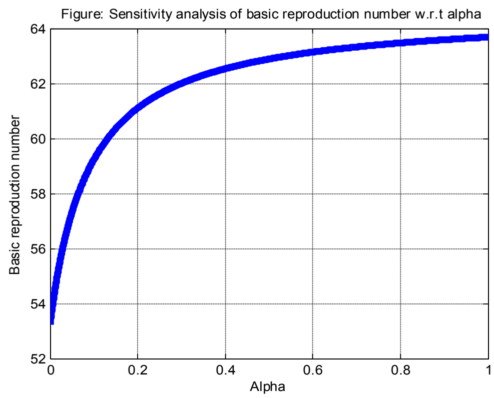
Substituting the above assumed parameters and initial conditions in the basic reproduction number we obtain the numerical value of the Basic Reproduction number as

$$R_0 = \frac{(\omega_3 + n\tau)\beta S^0}{\omega_2\omega_3 - \propto \tau} \approx 62$$

This means that the combined population of light users and addicted which totals to 1000 (990+10) would recruit approximately 62 other individuals using the next generation method.

4.2 Numerical sensitivity of the basic reproduction number

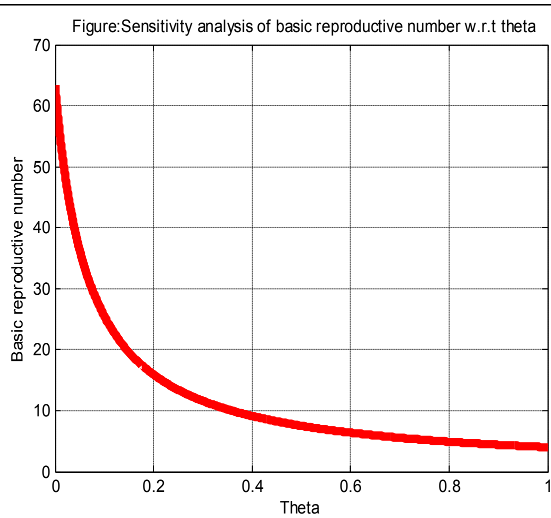
We use the assumed parameters to carry out the sensitivity analysis of the basic reproduction number and obtain the results represented graphically using MATLAB software as below:



Discussion

The basic reproduction number is directly proportional to the rate of quitting from addict to light users. This confirms our earlier analytical expression in section 4.2.9 that is $\frac{\partial R_0}{\partial \alpha} > 0$, which was not possible then. Therefore quitting from addict to light user may not reduce the burden and so people should be encouraged to quit completely.

Fig. 1. Sensitivity analysis of basic reproduction number w.r.t alpha



Discussion

The basic reproduction number is inversely proportional to the rate of quitting from light users to susceptible. This confirms our earlier analytical expression in section 4.2.9 that is $\frac{\partial R_0}{\partial \theta} < 0$ .when people quit from light users to susceptible, this is likely to reduce the burden of miraa abuse.

Fig. 2. Sensitivity analysis of basic reproduction number w.r.t theta

4.3 Normalized sensitivity analysis

The normalized forward sensitivity index of a variableR_0 that depends on differentiability of a parameter, X, is defined as $R_X^{R_0} = \frac{\partial R_0}{\partial X} \times \frac{X}{R_0}$ [21].

Sensitivity indices of R_0 to parameters of the Miraa model, are obtained in the Table 2 below:

Table 2. Sensitivity index of the model parameters

Parameters	**γ**	**μ**	**τ**	**θ**	**δ**	**η**	**α**	**β**
Sensitivity index	-0.7307	-0.2419	-0.03753	-0.0192	-0.00182	0.020039	0.0313	1

The parameters which have negative sensitivity index are inversely proportional to basic reproduction number (R_0) while those that have positive sensitivity index are directly proportional to basic reproduction number(R_0). Increase in values of parameter with negative sensitivity index and decrease in values of parameters with positive sensitivity index hold great promise in lowering impact of miraa abuse. Effort should also be made to determine actual values of the parameters in Kenya in order to obtain more realistic reproduction numbers. We may not have control over natural death rate μ and exit rate γ. Efforts should be made to increase θ.

4.4 Numerical simulation of the model

We used the assumed parameters to carry out model simulation using MATLAB software and obtained results represented graphically as below:

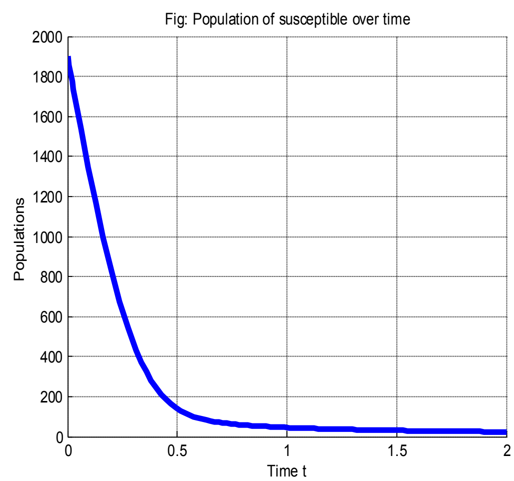
	Discussion
	The population of susceptible is likely to decrease with time when R_o is greater than 1 .

Fig. 3. Population of susceptible over time

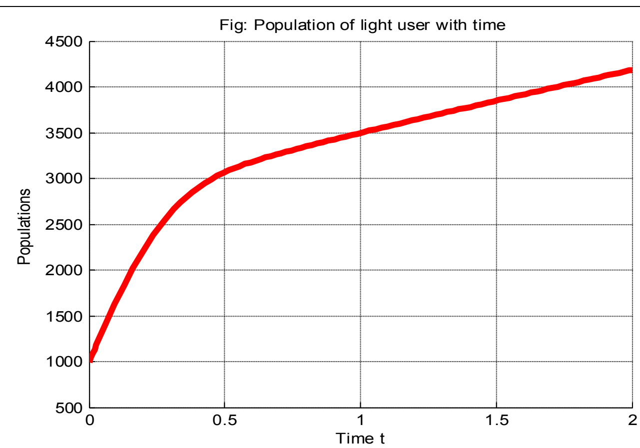
	Discussion
	The population of the light users is likely to increase with time when R_o is greater than 1.

Fig. 4. Population of light user with time

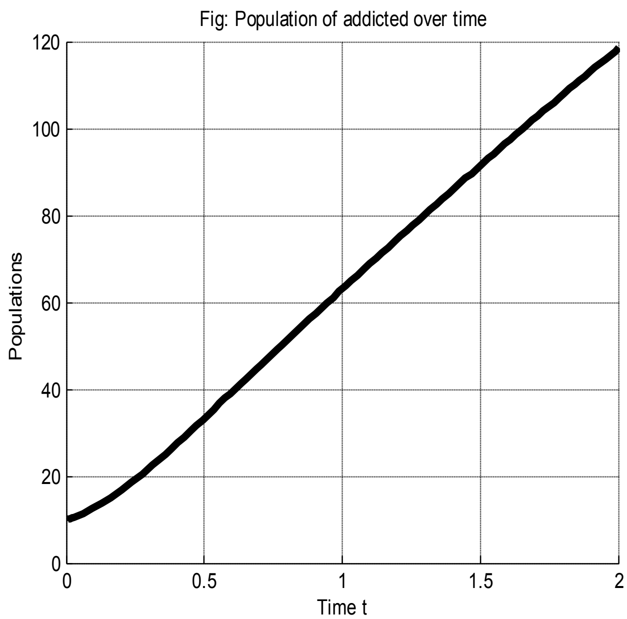
	Description
	The population of addicts increases with time when R_o is greater than 1..

Fig. 5. Population of addicted over time

5 Conclusion and Recommendations

The objective of the study was to develop a deterministic model that describes the dynamics of miraa addiction. From the model developed with three compartmental classes (S-L-A), quitting from light user to susceptible and from addict to susceptible group was incorporated. From the flow chart, three nonlinear first order ordinary differential equations that governed the dynamics of miraa addiction were deduced.

An expression for the basic reproduction number (R_0) was obtained. The analytical expression for (R_0) is important since it gives the average number of susceptible persons that one miraa consumer (light user or addict) can recruit in a susceptible population. Miraa Free Equilibrium point was found to exist and was locally asymptotically stable whenever $R_0 < 1$ and unstable otherwise. A positive Miraa Persistent Equilibrium point existed and was locally asymptotically stable whenever $R_0 > 1$.

The sensitivity analysis showed that the basic reproduction number is inversely proportional to the rate of quitting from light users to susceptible population. This implies that an increase in the number of people quitting from light users group reduces the rate of recruitment. In order to reduce the burden of miraa consumption therefore more efforts should be made to encourage light users to quit. People quitting from addict to light users may not provide a promising solution since this increases the rate of recruitment.

Further studies should be carried out to obtain reliable data and parameter estimates of people who are: susceptible, light users and addicted in order to obtain more reliable and valid results.

Competing Interests

Authors have declared that no competing interests exist.

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