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Research Article

Quantization of the Super-BMS$_3$ Algebra

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In this article, we study quantization of super-BMS$_3$ algebra W. We quantize W by the Drinfel'd twist quantization technique and obtain a class of noncommutative and noncocommutative Hopf superalgebras.

1. Introduction

Lie (super-)bialgebras as well as their quantizations provide important tools in searching for solutions of quantum Yang-Baxter equations and in producing new quantum groups [1, 2]. The notion of Lie bialgebras was introduced by Drinfel'd in 1983 [1, 3]. In 1992, the problem that whether there exists a general approach to quantize all Lie (super-)bialgebras was posed by Drinfel'd in [4]. Later, a positive answer was given by Etingof and Kazhdan in [5], but they did not present a uniform method to realize quantizations for all Lie (super-)bialgebras. Since then, the study of quantizations of Lie (super-)bialgebras has attracted more and more attention. A growing number of people studied the structure theory of Lie (super-)bialgebras, such as [6–11].

The "quantum group" appeared in the work of Drinfel'd as a deformation of the universal enveloping algebra of a Lie algebra in the category of Hopf algebras. In the theory of Hopf algebras and quantum groups, there exist two standard methods to yield new bialgebras from old ones. Twisting the product by a 2-cocycle but keeping the coproduct unchanged is one way; using a Drinfel'd twist element to twist the coproduct but preserving the product is the other approach. Constructing quantizations of Lie bialgebras is an important approach to producing new quantum groups [1, 2, 12].

As an application of quantum groups, quantizations of Lie (super-)bialgebra structures were intensively investigated. Recently, some authors have considered the quantization of several algebras, such as [11–18]. These algebras are all centerless. The case with center is similar.

In order to study the precise boundary conditions for the gauge field describing the theory, the super-BMS$_3$ algebra was introduced in [19, 20]. In [19], the author applies the construction to three-dimensional asymptotically flat $N = 1$ supergravity, whose algebra of surface charges has been shown to realize the centrally extended super-BMS$_3$ algebra. In this article, we study quantization of centerless super-BMS$_3$ algebra W.

The centerless super-BMS$_3$ algebra W is an infinite-dimensional Lie superalgebra over $\mathbb{C}$ with basis $\{L_m, P_m, Q_p \mid m \in \mathbb{Z}, p \in \varepsilon + \mathbb{Z}\}$ ($\varepsilon = 0$ or $1/2$) and satisfying the following relations:

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}, [L_m, P_n] = (m-n)P_{m+n},\\
&[L_m, Q_p] = \left(\frac{m}{2} - p\right)Q_{m+p}, [Q_p, Q_q] = P_{p+q},\\
&[P_m, P_n] = [P_m, Q_p] = 0,
\end{aligned} \tag{1}$$

for any $m, n \in \mathbb{Z}$ and $p, q \in \varepsilon + \mathbb{Z}$ ($\varepsilon = 0$ or $1/2$). The fermionic generators Q_p are labeled by (half-)integers in the case of (anti)periodic boundary conditions for the gravitino [19]. Clearly, the $\mathbb{Z}_2$-graded W is defined by $W = W_{\bar{0}} \oplus W_{\bar{1}}$, where $W_{\bar{0}} = \{L_m, P_m \mid m \in \mathbb{Z}\}$ and $W_{\bar{1}} = \{Q_p \mid p \in \varepsilon + \mathbb{Z}\}$. W contains the Witt algebra. The Lie super-bialgebra structures of W have been determined in [21]. The non weight modules of W have been studied in [20].

In this article, we study quantization of centerless super-BMS$_3$ algebra W. In Section 2, we use the general method of quantization by Drinfel'd twist to quantize explicitly the Lie bialgebra structures on W and obtain a family of noncommutative and noncocommutative Hopf superalgebras. The main result of the article is stated in Theorem 18.

2. Quantization of W

Theorem 1 (see [21]). *Every Lie super-bialgebra structure on W is a triangular coboundary Lie super-bialgebra.*

Proof. The super-BMS$_3$ algebra W in this article is the centerless case of that. By [21], we can deduce that every Lie super-bialgebra structure on W is a triangular coboundary Lie super-bialgebra. □

Definition 2 (see [11]). Let W be a Lie superalgebra containing linearly independent elements a and b satisfying $[a, b] = kb$ with $|a| = |b| = \bar{0}$ and $0 \neq k \in \mathbb{C}$. Then, we set $r = a \otimes b - b \otimes a$ and define a linear map $\delta = \delta_r : L \longrightarrow L \otimes L$ by requiring that

$$\delta_r(x) = (-1)^{|r||x|} x \cdot r = (-1)^{|r||x|}\left([x, a] \otimes b + (-1)^{|a||x|} a \otimes [x, b] - [x, b] \otimes a - \left(-1\right)^{|b||x|} b \otimes [x, a]\right) \tag{2}$$

for all $x \in W$. Then, δ_r equips W with the structure of a triangular coboundary Lie super-bialgebra.

Definition 3 (see [11]). A superalgebra (H, μ, ι) is a superspace H equipped with a unit $\iota : \mathbb{C} \longrightarrow H$, an associative product $\mu : H \otimes H \longrightarrow H$ respecting the grading, and the identity element $1 \in H_{\bar{0}}$. A Hopf superalgebra $(H, \mu, \iota, \Delta, S, \varepsilon)$ is a superalgebra (H, μ, ι) equipped with a coproduct $\Delta : H \longrightarrow H \otimes H$, a counit $\varepsilon : H \longrightarrow \mathbb{C}$, and an antipode $S : H \longrightarrow H$, satisfying certain compatible conditions. Note that the antipode S satisfies $S(xy) = (-1)^{|x||y|} S(y) S(x), \forall x, y \in H$.

Definition 4 (see [12, 15]). For any element x of a unital R-algebra (R is a ring) and $a \in R$, $r, k \in \mathbb{N}^+$, we set

$$\begin{aligned} x_a^{(r)} &= (x + a)(x + a + 1) \cdots (x + a + r - 1), \\ x_a^{[r]} &= (x + a)(x + a - 1) \cdots (x + a - r + 1), \\ \binom{a}{r} &= \frac{a(a-1) \cdots (a - r + 1)}{r!}, \\ \binom{a}{r}_k &= \frac{a(a-k) \cdots (a-(r-1)k)}{r!}. \end{aligned} \tag{3}$$

In particular, we set $x_0^{(r)} = x^{(r)}$, $x_0^{[r]} = x^{[r]}$, and $x_a^{(0)} = 1$, $x_a^{[0]} = 1$.

Lemma 5 (see [12, 15]). *For any element x of a unital $\mathbb{C}$-algebra and $a \in \mathbb{C}$, $r, s, t \in \mathbb{N}^+$, one has*

$$\begin{aligned} & x_a^{(s+t)} = x_a^{(s)} x_{a+s}^{(t)}, x_a^{[s+t]} = x_a^{[s]} x_{a+s}^{[t]}, x_a^{[s]} = x_{a-s+1}^{(s)}, \\ & \sum_{s+t=r} \frac{(-1)^t}{s!t!} x_a^{[s]} x_b^{(t)} = \binom{a-b}{r} = \frac{(a-b) \cdots (a-b-r+1)}{r!}, \\ & \sum_{s+t=r} \frac{(-1)^t}{s!t!} x_a^{[s]} x_{b-s}^{[t]} = \binom{a-b+r-1}{r} = \frac{(a-b) \cdots (a-b+r-1)}{r!}. \end{aligned} \tag{4}$$

Definition 6 (see [1]). Let $(H, \mu, \iota, \Delta_0, S_0, \varepsilon)$ be a Hopf superalgebra. A Drinfel'd twist $\mathcal{F}$ on H is an invertible element in $H \otimes H$ such that

$$\begin{aligned} (\mathcal{F} \otimes 1)(\Delta_0 \otimes Id)(\mathcal{F}) &= (1 \otimes \mathcal{F})(Id \otimes \Delta_0)(\mathcal{F}), (\varepsilon \otimes Id)(\mathcal{F}) \\ &= 1 \otimes 1 = (Id \otimes \varepsilon)(\mathcal{F}). \end{aligned} \tag{5}$$

Lemma 7 (see [22]). *Let $(H, \mu, \iota, \Delta_0, S_0, \varepsilon)$ be a Hopf superalgebra and $\mathcal{F}$ be a Drinfel'd twist on H. Then, $w = \mu(Id \otimes S_0)\mathcal{F}$ is invertible in H with $w^{-1} = \mu(Id \otimes S_0)\mathcal{F}^{-1}$. Moreover, we denote $\Delta : H \longrightarrow H \otimes H$ and $S : H \longrightarrow H$ by*

$$\begin{aligned} \Delta(x) &= \mathcal{F}\Delta_0(x)\mathcal{F}^{-1}, \\ S(x) &= wS_0(x)w^{-1}, \quad \forall x \in H. \end{aligned} \tag{6}$$

Then, $(H, \mu, \iota, \Delta, S, \varepsilon)$ is a new Hopf superalgebra, which is called twisting of H by the Drinfel'd twist $\mathcal{F}$.

Lemma 8 (see [12, 15]). *For any elements x, y in an associative algebra, $p \in \mathbb{N}$, one has*

$$xy^p = \sum_{k=0}^{p} (-1)^k \binom{p}{k} y^{p-k} (ady)^k (x). \tag{7}$$

Definition 9. Let $U(W)$ be the universal enveloping algebra of W and $(U(W), \mu, \iota, \Delta_0, S_0, \varepsilon)$ be the standard Hopf algebra structure on $U(W)$. Then, the coproduct Δ_0, the antipode S_0, and the counit ε are defined by

$$\begin{aligned} \Delta_0(X) &= X \otimes 1 + 1 \otimes X, \\ S_0(X) &= -X, \\ \varepsilon(X) &= 0, \quad \forall X \in W. \end{aligned} \tag{8}$$

In particular, $\Delta_0(1) = 1 \otimes 1$ and $S_0(1) = \varepsilon(1) = 1$.

Lemma 10. *Let $X = -(1/m)L_0$ and $Y = P_m$ ($m \in \mathbb{Z}^*$); we have $[X, Y] = Y$; then, X and Y generate a two-dimensional nonabelian subalgebra of W.*

Proof. For any $m \in \mathbb{Z}^*$, by $[L_0, P_m] = -mP_m$, we can get $[X, Y] = Y$. Then, we set $r = X \otimes Y - Y \otimes X$. By Definition

2, δ_r equips W with the structure of a triangular coboundary Lie super-bialgebra. □

Lemma 11. *For any $a \in \mathbb{C}$, $n \in \mathbb{Z}$, $p \in \varepsilon + \mathbb{Z}$, $m \in \mathbb{Z}^*$, and $r \in \mathbb{N}^+$, we have*

$$L_n X_a^{(r)} = X_{a-(n/m)}^{(r)} L_n, L_n X_a^{[r]} = X_{a-(n/m)}^{[r]} L_n, \tag{9}$$

$$P_n X_a^{(r)} = X_{a-(n/m)}^{(r)} P_n, P_n X_a^{[r]} = X_{a-(n/m)}^{[r]} P_n, \tag{10}$$

$$Q_p X_a^{(r)} = X_{a-(n/m)}^{(r)} Q_p, Q_p X_a^{[r]} = X_{a-(n/m)}^{[r]} Q_p. \tag{11}$$

Proof. We only prove that (9), (10), and (11) can be obtained similarly. We prove (9) by induction on r. It is true for the case of $r = 1$. Assume the case of r is also true; then, we consider the case of $r + 1$; we have

$$L_n X_a^{(r+1)} = L_n X_a^{(r)} (X + a + r) = X_{a-(n/m)}^{(r)} L_n (X + a + r), \tag{12}$$

$$[X, L_n] = -\frac{1}{m} [L_0, L_n]. \tag{13}$$

By (12) and (13), we have

$$\begin{aligned} X_{a-(n/m)}^{(r)} L_n (X + a + r) &= X_{a-(n/m)}^{(r)} \left(X L_n - \frac{n}{m} L_n + (a + r) L_n \right) \\ &= X_{a-(n/m)}^{(r)} \left(X + a - \frac{n}{m} + r \right) L_n \\ &= X_{a-(n/m)}^{(r+1)} L_n. \end{aligned} \tag{14}$$

Therefore, we deduce that $L_n X_a^{(r+1)} = X_{a-(n/m)}^{(r+1)} L_n$, which means that $L_n X_a^{(r)} = X_{a-(n/m)}^{(r)} L_n$ is true. The proof of $L_n X_a^{[r]} = X_{a-(n/m)}^{[r]} L_n$ is similar. □

Lemma 12. *For any $a \in \mathbb{C}$, $s, r \in \mathbb{N}^+$, we have*

$$Y^s X_a^{(r)} = X_{a-s}^{(r)} Y^s, Y^s X_a^{[r]} = X_{a-s}^{[r]} Y^s. \tag{15}$$

Proof. The case of $s = r = 1$ is clear. If $s = 1$, we prove (15) by induction on r. We have

$$\begin{aligned} Y X_a^{(r+1)} &= Y X_a^{(r)} (X + a + r) = X_{a-1}^{(r)} Y (X + a + r) \\ &= X_{a-1}^{(r)} (XY - Y + (a + r) Y) \\ &= X_{a-1}^{(r)} (X + a + r - 1) Y = X_{a-1}^{(r+1)} Y, \end{aligned} \tag{16}$$

which means that

$$Y X_a^{(r)} = X_{a-1}^{(r)} Y. \tag{17}$$

Suppose that $Y^s X_a^{(r)} = X_{a-s}^{(r)} Y^s$. By (17), we have

$$Y^{s+1} X_a^{(r)} = Y X_{a-s}^{(r)} Y^s = X_{a-(s+1)}^r Y Y^s = X_{a-(s+1)}^r Y^{s+1}. \tag{18}$$

The proof of $Y^s X_a^{[r]} = X_{a-s}^{[r]} Y^s$ is similar. □

Lemma 13. *For any $m \in \mathbb{Z}^*$, $n \in \mathbb{Z}$, $p \in \varepsilon + \mathbb{Z}$, and $r \in \mathbb{N}^+$, we have*

$$L_n Y^r = Y^r L_n + r Y^{r-1} (n - m) P_{m+n}, \tag{19}$$

$$P_n Y^r = Y^r P_n, \tag{20}$$

$$Q_p Y^r = Y^r Q_p. \tag{21}$$

Proof. We only prove (19) and (21). The proof of (20) is similar.

$$\begin{aligned} L_n Y^r &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} Y^{r-k} (adY)^k L_n = Y^r L_n + r Y^{r-1} (n - m) P_{m+n}, \\ Q_p Y^r &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} Y^{r-k} (adY)^k Q_p = Y^r Q_p. \end{aligned} \tag{22}$$

□

Definition 14 (see [12, 15]). For $a \in \mathbb{C}$, set

$$\begin{aligned} \mathcal{F}_a &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_a^{[r]} \otimes Y^r t^r, \\ F_a &= \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes Y^r t^r, \end{aligned} \tag{23}$$

$$\begin{aligned} U_a &= \mu (S_0 \otimes I_d) F_a, \\ V_a &= \mu (I_d \otimes S_0) \mathcal{F}_a. \end{aligned} \tag{24}$$

In particular, we set $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, and $v = v_0$. Since $S_0 (X_a^{(r)}) = (-1)^r X_{-a}^{[r]}$ and $S_0 (Y^r) = (-1)^r Y^r$, we have

$$U_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a}^{[r]} Y^r t^r V_a = \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{[r]} Y^r t^r. \tag{25}$$

Lemma 15. *For $a, b \in \mathbb{C}$, we have $\mathcal{F}_a F_b = 1 \otimes (1 - Yt)^{a-b}$ and $V_a U_b = (1 - Yt)^{-a-b}$; $\mathcal{F}_a$, F_a, U_a, and V_a are invertible elements with $\mathcal{F}_a = F_a^{-1}$ and $U_a = V_a^{-1}$.*

Proof. By (23) and Lemma 5, we have

$$\begin{aligned}\mathcal{F}_a F_b &= \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r!s!} X_a^{[r]} X_b^{(s)} \otimes Y^r Y^s t^r t^s \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{r+s=k} \frac{(-1)^s}{r!s!} X_a^{[r]} X_b^{(s)} \otimes Y^k t^k \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{a-b}{k} \otimes Y^k t^k = 1 \otimes (1 - Yt)^{a-b}.\end{aligned} \tag{26}$$

From (15), (25), and Lemma 5, we obtain

$$\begin{aligned}V_a U_b &= \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r!s!} X_a^{[r]} Y^r X_{-b}^{[s]} Y^s t^{r+s} \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{r+s=k} \frac{(-1)^s}{r!s!} X_a^{[r]} X_{-b-r}^{[s]} Y^k t^k \\ &= \sum_{k=0}^{\infty} \binom{a+b+k-1}{k} Y^k t^k = (1 - Yt)^{-a-b}.\end{aligned} \tag{27}$$

Then, we can deduce that $\mathcal{F}_a = F_a^{-1}$, $U_a = V_a^{-1}$, $\mathcal{F} = F^{-1}$, and $U = V^{-1}$. □

Lemma 16. *For any $m \in \mathbb{Z}^*$, $n \in \mathbb{Z}$, $p \in \varepsilon + \mathbb{Z}$, and $a \in \mathbb{C}$, we have*

$$(L_n \otimes 1) F_a = F_{a-(n/m)} (L_n \otimes 1), \tag{28}$$

$$(P_n \otimes 1) F_a = F_{a-(n/m)} (P_n \otimes 1), \tag{29}$$

$$\left(Q_p \otimes 1\right) F_a = F_{a-(n/m)} \left(Q_p \otimes 1\right), \tag{30}$$

$$(1 \otimes L_n) F_a = F_a (1 \otimes L_n) + (n - m) F_{a+1} \left(X_a^{(1)} \otimes P_{m+n} t\right), \tag{31}$$

$$(1 \otimes P_n) F_a = F_a (1 \otimes P_n), \tag{32}$$

$$\left(1 \otimes Q_p\right) F_a = F_a \left(1 \otimes Q_p\right). \tag{33}$$

Proof. We only prove (28), (31), and (32); the proof of other equations is similar. □

By (9) and (23), we have

$$\begin{aligned}(L_n \otimes 1) F_a &= \sum_{r=0}^{\infty} \frac{1}{r!} L_n X_a^{(r)} \otimes Y^r t^r = \sum_{r=0}^{\infty} \frac{1}{r!} X_{a-(n/m)}^{(r)} L_n \otimes Y^r t^r \\ &= F_{a-(n/m)} (L_n \otimes 1).\end{aligned} \tag{34}$$

The proof of (29)–(30) is similar to (28).

By (19) and (23), we have

$$\begin{aligned}(1 \otimes L_n) F_a &= \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes \left(Y^r L_n + r Y^{r-1} (n-m) P_{m+n}\right) t^r \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes Y^r L_n t^r + (n-m) \sum_{r=1}^{\infty} \frac{1}{(r-1)!} X_a^{(r)} \otimes Y^{r-1} P_{m+n} t^r \\ &= \left(\sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes Y^r t^r\right) (1 \otimes L_n) + (n-m) \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r+1)} \otimes Y^r P_{m+n} t^{r+1} \\ &= F_a (1 \otimes L_n) + (n-m) \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(1)} X_{a+1}^{(r)} \otimes Y^r P_{m+n} t^r \cdot t \\ &= F_a (1 \otimes L_n) + (n-m) \left(\sum_{r=0}^{\infty} \frac{1}{r!} X_{a+1}^{(r)} \otimes Y^r t^r\right) \left(X_a^{(1)} \otimes P_{m+n} t\right) \\ &= F_a (1 \otimes L_n) + (n-m) F_{a+1} \left(X_a^{(1)} \otimes P_{m+n} t\right).\end{aligned} \tag{35}$$

By (21) and (25), we have

$$\begin{aligned}\left(1 \otimes Q_p\right) F_a &= \left(1 \otimes Q_p\right) \left(\sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes Y^r t^r\right) = \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes Q_p Y^r t^r \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} X_a^{(r)} \otimes Y^r Q_p t^r = F_a \left(1 \otimes Q_p\right).\end{aligned} \tag{36}$$

Lemma 17. *For any $m \in \mathbb{Z}^*$, $n \in \mathbb{Z}$, $p \in \varepsilon + \mathbb{Z}$, and $a \in \mathbb{C}$, we have*

$$L_n U_a = U_{a+(n/m)} L_n + (m - n) U_{a+(n/m)} X_{-a-(n/m)}^{[1]} P_{m+n} t, \tag{37}$$

$$P_n U_a = U_{a+(n/m)} P_n, \tag{38}$$

$$Q_p U_a = U_{a+(n/m)} Q_p. \tag{39}$$

Proof. We only prove (37) and (39); the case of (38) is similar. By (9), (11), (15), (19), and (24), we have

$$\begin{aligned}L_n U_a &= L_n \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a}^{[r]} Y^r t^r\right) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} L_n X_{-a}^{[r]} Y^r t^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} L_n Y^r t^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} \left(Y^r L_n + r Y^{r-1} (n-m) P_{m+n}\right) t^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} Y^r L_n t^r + \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!} X_{-a-(n/m)}^{[r]} Y^{r-1} (n-m) P_{m+n} t^r \\ &= U_{a+(n/m)} L_n + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{r!} X_{-a-(n/m)}^{[r+1]} Y^r (n-m) P_{m+n} t^{r+1} \\ &= U_{a+(n/m)} L_n + (m-n) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} X_{-a-(n/m)-r}^{[1]} Y^r P_{m+n} t^r \cdot t \\ &= U_{a+(n/m)} L_n + (m-n) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} Y^r X_{-a-(n/m)}^{[1]} P_{m+n} t^r \cdot t \\ &= U_{a+(n/m)} L_n + (m-n) U_{a+(n/m)} X_{-a-(n/m)}^{[1]} P_{m+n} t,\end{aligned}$$

$$
\begin{aligned}
Q_n U_a &= Q_n \left(\sum_{r=0}^{\infty} \frac{(-1^r)}{r!} X_{-a}^{[r]} Y^r t^r \right) = \sum_{r=0}^{\infty} \frac{(-1^r)}{r!} Q_n X_{-a}^{[r]} Y^r t^r \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} Q_n Y^r t^r = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X_{-a-(n/m)}^{[r]} Y^r Q_n t^r \\
&= U_{a+(n/m)} Q_n.
\end{aligned} \tag{40}
$$

□

Theorem 18. *With the choice of two distinguished elements $X = -(1/m)L_0$ and $Y = P_m$ ($m \in \mathbb{Z}^*$) such that $[X, Y] = Y$ in W, there exists a structure of noncommutative algebra and noncocommutative Hopf algebra structure $(U(W)[[t]], \mu, \iota, \Delta, S, \varepsilon)$ on $U(W)[[t]]$, such that $U(W)[[t]]/tU(W)[[t]] \cong U(W)$, which preserve the product and counit of $U(W)[[t]]$; the coproduct and antipode are defined by*

$$
\Delta(L_n) = L_n \otimes (1 - Yt)^{n/m} + 1 \otimes L_n + (n-m)\left(X^{(1)} \otimes (1 - Yt)^{-1} P_{m+n} t \right), \tag{41}
$$

$$
\Delta(P_n) = P_n \otimes (1 - Yt)^{n/m} + 1 \otimes P_n, \tag{42}
$$

$$
\Delta\left(Q_p\right) = Q_p \otimes (1 - Yt)^{n/m} + 1 \otimes Q_p, \tag{43}
$$

$$
S(L_n) = -(1 - Yt)^{-n/m} \left(L_n + (m-n) X_{-(n/m)}^{[1]} P_{m+n} t \right), \tag{44}
$$

$$
S(P_n) = -(1 - Yt)^{-(n/m)} P_n, \tag{45}
$$

$$
S\left(Q_p\right) = -(1 - Yt)^{-(n/m)} Q_p, \tag{46}
$$

where $m \in \mathbb{Z}^*$, $n \in \mathbb{Z}$, and $p \in \varepsilon + \mathbb{Z}$.

Proof. We only prove (41), (43), (44), and (46); the cases of (42) and (45) are similar. By Definition 14 and Lemmas 15–17, we have

$$
\begin{aligned}
\Delta(L_n) &= \mathcal{F}\Delta_0(L_n)\mathcal{F}^{-1} = \mathcal{F}(L_n \otimes 1 + 1 \otimes L_n)\mathcal{F}^{-1} \\
&= \mathcal{F}(L_n \otimes 1)F + \mathcal{F}(1 \otimes L_n)F \\
&= \mathcal{F}F_{-(n/m)}(L_n \otimes 1) + \mathcal{F}\left(F(1 \otimes L_n) + (n-m)F_1\left(X^{(1)} \otimes P_{m+n} t \right) \right) \\
&= \left(1 \otimes (1 - Yt)^{n/m}\right)(L_n \otimes 1) + 1 \otimes L_n + (n-m) \\
&\quad \cdot \left(X^{(1)} \otimes (1 - Yt)^{-1} P_{m+n} t \right) = L_n \otimes (1 - Yt)^{n/m} + 1 \otimes L_n \\
&\quad + (n-m)\left(X^{(1)} \otimes (1 - Yt)^{-1} P_{m+n} t \right),
\end{aligned}
$$

$$
\begin{aligned}
\Delta(Q_n) &= \mathcal{F}\Delta_0(Q_n)\mathcal{F}^{-1} = \mathcal{F}(Q_n \otimes 1 + 1 \otimes Q_n)\mathcal{F}^{-1} \\
&= \mathcal{F}(Q_n \otimes 1)F + \mathcal{F}(1 \otimes Q_n)F \\
&= \mathcal{F}F_{-(n/m)}(Q_n \otimes 1) + \mathcal{F}F(1 \otimes Q_n) \\
&= Q_n \otimes (1 - Yt)^{n/m} + 1 \otimes Q_n,
\end{aligned}
$$

$$
\begin{aligned}
S(L_n) &= -VL_nU = -V\left(U_{n/m} L_n + (m-n) U_{n/m} X_{-(n/m)}^{[1]} P_{m+n} t \right) \\
&= -VU_{n/m}\left(L_n + (m-n) X_{-(n/m)}^{[1]} P_{m+n} t \right) \\
&= -(1 - Yt)^{-(n/m)} \left(L_n + (m-n) X_{-(n/m)}^{[1]} P_{m+n} t \right),
\end{aligned}
$$

$$
S(Q_n) = -VQ_nU = -VU_{n/m}Q_n = -(1 - Yt)^{-(n/m)} Q_n. \tag{47}
$$

□

Remark 19. We can use the method in this paper to study the quantization of other Lie (super-)algebras in the future.

Remark 20. The case with center is similar, because the center element C can be exchanged. Namely, $[C, W] = 0$.

Data Availability

No data is available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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