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Reevaluating the Complementary Relationship between Single Ultrafilters and Linear Obstacles in Connectivity Systems

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The study of graph width parameters is highly significant in graph theory and combinatorics. Among these parameters, linear-width is particularly well-regarded and established. The concepts of Single Filter and Linear Obstacle pose challenges to achieving optimal linear-width in a connectivity system. In this concise paper, we present an alternative proof that establishes the cryptomorphism between Single Filter and Linear Obstacle. Although this proof may not be highly novel, we hope it will enhance the understanding of the intricate relationship between graph width parameters and ultrafilters.

Keywords: Linear width; single filter; linear obstacle; connectivity system.

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1 Introduction

The "graph width parameter" is a metric in graph theory that measures the width of a graph. It generally represents the maximum width across all cuts or layers in a hierarchical decomposition of the graph. The study of width parameters is pivotal in graph theory and combinatorics, as evidenced by the extensive literature on this topic (e.g., [1-6,7-12,13-26]).

One of width parameter is tree-width and branch-width. Branch-width and tree-width are two important concepts in graph theory that measure different aspects of a graph's complexity. Each provides a framework for understanding how a graph can be decomposed, which has applications in algorithm design and other areas of theoretical computer science. Similarly, linear width, a specialized variant of branch-width, has also been extensively explored. Thus, the investigation into both branch-width and linear width is of critical importance.

The concept of a Single filter, introduced in reference [3], serves as a modeling tool for the mathematical "filter" in Boolean algebra and topology. Within a connectivity system, the Single filter acts as the counterpart to linear width, a relationship that is also discussed. Connectivity systems provide a mathematical framework primarily used in graph theory and combinatorial optimization to analyze the connectivity properties of structures such as graphs, networks, and matroids. Additionally, the concept of a linear obstacle in a connectivity system is identified as another counterpart to linear width [6,22].

In this paper, we explore the cryptomorphism result between single filters and linear obstacles. While this proof may not be highly novel, we aim to enhance the understanding of the intricate relationship between graph width parameters and ultrafilters [27,28].

2 Definitions in This Paper

This section provides mathematical definitions for each concept. In this short paper, we use expressions like $A \subseteq X$ to indicate that A is a subset of X, $A \cup B$ to represent the union of two subsets A and B, both of which are subsets of X, or $A = \emptyset$ to signify an empty set. Specifically, $A \cap B$ denotes the intersection of subsets A and B. A similar logic applies to $A \setminus B$.

2.1 Symmetric submodular function and connectivity system

A symmetric submodular function is a set function characterized by both submodularity and symmetry, meaning it exhibits diminishing returns and remains unchanged when elements are interchanged. This function is widely used in combinatorial optimization, particularly in applications such as clustering, facility location, and network design. Although symmetric submodular functions typically assume real values, this paper specifically focuses on those functions restricted to natural numbers. The definition of a symmetric submodular function is provided below [29].

Definition 2.1.1

Let X be a finite set. A function $f: X \to \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

$\forall A \subseteq X, f(A) = f(X \setminus A).$ $\forall A, B \subseteq X, f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$

To illustrate the use of the function and provide a clear understanding, here is an example.

Example 2.1.2

Consider $X = \{a, b, c\}$ and the function $f: X \to \mathbb{N}$ defined as follows:

$f(\emptyset) = 0, f(\{a\}) = f(\{b\}) = f(\{c\}) = 1, f(\{a, b\}) = f(\{a, c\}) = f(\{b, c\}) = 2, f(\{a, b, c\}) = 1$

This function is symmetric because $f(A) = f(X \setminus A)$ for all $A \subseteq X$.

It is also submodular because for any $A, B \subseteq X$, the inequality $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ holds. For example, let $A = \{a\}$ and $B = \{b, c\}$:

$$f(A) + f(B) = f(\{a\}) + f(\{b, c\}) = 1 + 2 = 3, f(A \cap B) + f(A \cup B) = f(\emptyset) + f(\{a, b, c\}) = 0 + 1 = 1$$

Since $3 \geq 1$, the function satisfies the submodularity condition. ■

A symmetric submodular function possesses the following properties. This lemma will be utilized in the proofs of lemmas and theorems presented in this paper.

Lemma 2.1.3 [7]

A symmetric submodular function f satisfies:

1. $\forall A \subseteq X$, $f(A) \geq f(\emptyset) = f(X) = 0$,

2. $\forall A, B \subseteq X$, $f(A) + f(B) \geq f(A \backslash B) + f(B \backslash A)$.

In this brief paper, a connectivity system is defined as a pair *(X, f)* consisting of a finite set (an underlying set) X and a symmetric submodular function *f*. And, throughout this paper, we use the notation *f* to refer to a symmetric submodular function, a finite set (an underlying set) *X*, and natural numbers *k*, *m*. A set *A* is said to be *k*-efficient if $f(A) \leq k$ [30].

2.2 Single filter on a connectivity system *(X,f)*

The definition of a single filter on a connectivity system *(X,f)* is given below. The concept of a single filter is useful when discussing linear branch-decomposition.

Definition 2.2.1 [3]

Let *X* represent a finite set and *f* denote a symmetric submodular function. A subset $S \subseteq 2^X$ is called an order $k + 1$ single ultrafilter on a connectivity system *(X, f)* if it satisfies the following conditions:

(S1) For any $A \in S$, $e \in X$, if $f(\{e\}) \leq k$ and $f(A \cap (X/\{e\})) \leq k$, then $A \cap (X/\{e\}) \in S$.

(S2) For any $A \in S$ and $A \subset B \subseteq X$, if $f(B) \leq k$, then $B \in S$.

(F3) $\emptyset$ is not belong to *S*.

(S4) For any $A \subseteq X$, if $f(A) \leq k$, either $A \in S$ or $(X/A) \in S$.

And single ultrafilter is non-principal if the single ultrafilter satisfies following axiom:

(F5) $A \notin S$ for all $A \subseteq X$ with $|A| = 1, f(A) \leqq k$.

2.3 Linear obstacle on a connectivity system *(X,f)*

The definition of Linear obstacle is shown below. This concept is deep relation to *(k,m)*-obstacle in literature [6]. And the concept of a linear obstacle is useful when discussing linear branch-decomposition.

Definition 2.3.1 [22]

Let *X* represent a finite set and *f* denote a symmetric submodular function. In a connectivity system *(X,f)*, the set family $O \subseteq 2^X$ is called a linear obstacle of order $k + 1$ if the following axioms hold true:

(O1) $A \in O, f(A) \leq k$,

(O2) $A \subseteq B \subseteq X, B \in O, f(A) \leq k \Rightarrow A \in O,$

(O3) $A, B, C \subseteq X, A \cup B \cup C = X, A \cap B = \emptyset, f(A) \leq k, f(B) \leq k, |C| \leq 1 \Rightarrow$ either $A \in O$ or $B \in O$.

3 Results: Cryptomorphism between Single Filter and Linear Obstacle

The result of this short paper is below. This theorem demonstrates the cryptomorphism between a single filter and a linear obstacle.

Theorem 3.1

Let X represent a finite set and f denote a symmetric submodular function. Assuming that $f(\{e\}) \leq k$ for every $e \in X$, S is a single ultrafilter of order $k+1$ on (X,f) if and only if O={A| X\A∈$S\}$ is a linear obstacle of *order* $k+1$ *on* (X,f).

Proof: To prove this theorem, we need to show two implications:

- If S is a single ultrafilter of order $k+1$, then $O = \{A \mid X \setminus A \in S\}$ is a linear obstacle of order $k+1$.
- If $O = \{A \mid X \setminus A \in S\}$ is a linear obstacle of order $k+1$, then S is a single ultrafilter of order $k+1$.

(1) Single Ultrafilter to Linear Obstacle:

Assume S is a single ultrafilter of order k+1. We need to show that $O = \{A \mid X \setminus A \in S\}$ satisfies the axioms (O1), (O2), and (O3) of a linear obstacle.

Condition (O1): Let $A \in O$ and $f(A) \leq k$. By definition, $A \in O$ means $X \setminus A \in S$. Since $f(A) \leq k$, $f(X \setminus A) = f(A) \leq k$, which by the definition of S being a single ultrafilter means $X \setminus (X \setminus A) = A \in S$, ensuring $A \in O$.

Condition (O2): Let $A \subseteq B \subseteq X$, $B \in O$, and $f(A) \leq k$. Since $B \in O$, $X \setminus B \in S$. Given $f(A) \leq k$, $f(X \setminus A) \leq k$. Because $X \setminus A \supseteq X \setminus B$ and $X \setminus B \in S$, by condition (S2), $X \setminus A \in S$, hence $A \in O$.

Condition (O3): Let $A, B, C \subseteq X$, $A \cup B \cup C = X$, $A \cap B = \emptyset$, $f(A) \leq k$, $f(B) \leq k$, and $|C| \leq 1$. We need to show that either $A \in O$ or $B \in O$. Assume for contradiction that $A \notin O$ and $B \notin O$. Then $X \setminus A \notin S$ and $X \setminus B \notin S$. Since $f(A) \leq k$ and $f(B) \leq k$, $f(X \setminus (X \setminus A)) = f(A) \leq k$ and $f(X \setminus (X \setminus B)) = f(B) \leq k$, by condition *(S4)* of the single ultrafilter, either $X \setminus A \in S$ or $X \setminus B \in S$. This leads to a contradiction. Therefore, either $A \in O$ or $B \in O$.

Hence, O satisfies all conditions for being a linear obstacle of order $k+1$.

(2) Linear Obstacle to Single Ultrafilter:

Assume $O = \{A \mid X \setminus A \in S\}$ is a linear obstacle of order $k+1$. We need to show that S satisfies the axioms (S1), (S2), (S3), and (S4) of a single ultrafilter.

Condition (S1): Let $A \in S$ and $e \in X$ such that $f(\{e\}) \leq k$ and $f(A \cap (X \setminus \{e\})) \leq k$. Since $A \in S$, $X \setminus A \in O$. Given $f(A \cap (X \setminus \{e\})) \leq k$, $f(X \setminus (A \cap (X \setminus \{e\}))) = f((X \setminus A) \cup \{e\}) \leq k$. By condition (O2), since $(X \setminus A) \cup \{e\} \in O$, $X \setminus (A \cap (X \setminus \{e\})) \in S$, so $A \cap (X \setminus \{e\}) \in S$.

Condition (S2): Let $A \in S$ and $A \subseteq B \subseteq X$ such that $f(B) \leq k$. Since $A \in S$, $X \setminus A \in O$. Given f$(B) \leq k$, $f(X \setminus B) \leq k$. Since $X \setminus B \subseteq X \setminus A$ and $X \setminus A \in O$, by condition (O2), $X \setminus B \in O$. Therefore, $B \in S$.

Condition (S3): By definition, $\emptyset \notin S$.

Condition (S4): For any $A \subseteq X$, if $f(A) \leq k$, either $A \in S$ or $X \setminus A \in S$. Since $O = \{A \mid X \setminus A \in S\}$, and $f(A) \leq k$, by

Condition (O3), either $A \in S$ or $(X \setminus A) \in S$.

Thus, *S* satisfies all conditions for being a single ultrafilter of order *k+1*.Since both implications are true, the theorem is proven.

4 Conclusion

We proved the cryptomorphism result between single-filter and linear obstacle. We plan to continue examining the algebraic characteristics of single filters in the future.

Disclaimer

This paper is an extended version of a preprint document of the same author.
The preprint document is available in this link:
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Competing Interests

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