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# Some Operations and Their Closure Properties on Multiset Topological Spaces

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

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## Abstract

In this paper, we introduced some operations on multiset topological spaces to include union, intersection, arithmetic multiplication, Scalar multiplication and Raising to Arithmetic Power. Studies revealed that these operations are closed on the various classes of multiset topological spaces defined.

*Keywords:* *Multiset; topological space; arithmetic multiplication; scalar multiplication; raising to arithmetic power.*

## 1 Introduction

The notion of multiset (mset, for short) is well established both in mathematics and in computer science. In mathematics, a mset is considered to be the generalization of a set. In classical set theory, a set is a well-defined

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collection of distinct objects, if repeated occurrences of any object are allowed in a set, then a mathematical structure, that is known as mset is obtained. For the various applications of msets the reader is referred to article [1]. It is observed from the survey of available literature on msets and applications that the idea of mset was hinted by R. Dedikind in 1888. The mset theory which generalizes set theory as a special case was introduced by Cerf et al. [2]. The term mset, as noted by Knuth [3] was first suggested by N.G de Bruijn in a private communication to him. Further study was carried out by Yager [4], Blizard [5]. Other researchers [6,7,8] gave a new dimension to the multiset theory.

Research on the mset theory has been gaining grounds. The research carried out so far shows a strong analogy in the behaviour of sets and msets. It is possible to extend some of the main notion and result of sets to the setting of msets. The concept of mset topological space (M-topological space, for short) and the concept of open mset were introduced by Gerish and John [9]. More precisely, an M-topology is defined as a set of mset points. Furthermore, the notion of basis, sub basis, closed sets, closure and interior in topological spaces were extended to M-topological spaces and many related theorems also proved.

Mahanta and Das [10] studied the Semi Compactness in mset Topological space, by considering their properties. They introduce the concepts of semi open and semi closed msets in M-topological spaces. With this generalization of the notions of open and closed sets in M-topological space, they generalize the concept of compactness in M-topological space as semi compactness. Furthermore, semi compactness is generalized as semi whole compactness, semi partial whole compactness and semi full compactness. They also studied some of the characterizations of these compact spaces in the setting of mset theory. Furthermore, Mahanta, [11] introduced the concepts of exterior and boundary in mset topological space. They further established a few relationships between the concepts of boundary, closure, exterior and interior of an M- set. These concepts have been pigeonholed by other existing notions viz., open sets, closed sets, and limit points. The necessary and sufficient condition for an mset to have an empty exterior is also presented and justified.

Sobhy. A. El-Sheikh et al [12] extended the notion of Hausdorff topological space as T2 space, as part of the separation axioms where the set upon which the topology is built is a mset and introduced, studied the notion of mset bitopological spaces (M-bitopological space, for short). They further presented the notion of ij-pre-open msets, ij-$\alpha$-open msets, ij-semi-open msets, and ij-$\beta$-open msets. In recent years, neutrosophic soft bitopological spaces have emerged as a promising framework for handling uncertainty and imprecision in various domains, particularly in the context of decision-making problems. Hasan Dadas, [13] presents a comprehensive study of advanced decision-making techniques using generalized closed sets in neutrosophic soft bitopological spaces.

In this paper, we extended the concept of M-bitopological space to Hausdorff M-bitopological space and introduced operations on the various classes M-topological spaces to include submultispaces (submspace), union, intersection,arithmetic multiplication, scalar multiplication and raising to arithmetic power and established that these operations are closed on the various classes [14-16]. Consequently, we presented some basic definitions and notations in section 2. In section 3, our results are presented and the summary of our findings are given in section 4.

# 2 Basic Definitions and Notations

**Definition 2.1** Jena et al. [17] (mset): A mset $A$ drawn from the set $X$ is represented by a count function $m_A$ or $C_A$ defined as $C_A: X \longrightarrow N$, where $N$ is the set of non-negative integers.

Here $C_A(x)$ is the number of occurrences of the element $x$ in the mset $A$. The number $C_A(x)$ is assumed unique and finite from known areas of applications.We present the mset $A$ drawn from the set $X = \{x_1, x_2, x_3, \dots, x_n\}$ as $A = \{m_1/x_1, m_2/x_2, m_3/x_3, \dots, m_n/x_n\}$ where $m_i = C_A(x_i)\ \ i = 1,2,3, \dots, n$ in the mset $A$. However, those elements which are not included in the mset $A$ have zero count. i.e $C_A(x) = 0 \Leftrightarrow x \notin A$ and $C_A(x) > 0 \Leftrightarrow x \in A$. A mset $M$ is called simple or singleton if and only if for any $x, y \in M$, we have $x = y$. For example $M = [k/x]$ where $k > 0$. The mset $M$ over the set $X$ is said to be empty iff $C_M(x) = 0\ \forall\ x \in X$. We denote the empty mset by $\emptyset$ and $C_\emptyset(x) = 0, \forall\ x \in X$.

**Definition 2.2:** Let $A$ be a mset drawn from the set $X$. The root(support ) set of $A$ denoted $A^*$ is defined: $A^* = \{x \in X: C_A(x) > 0\}$. Note that $x \in A^* \Leftrightarrow x \in A$ for all $x$.

**Definition 2.3** Singh et al. [1] (Cardinality of an mset): The cardinality of an mset $A$ denoted $|A|$is the sum of the multiplicities of all the elements in $A$ i.e $|A| = \sum_{x \in X} C_A(x)$

**Definition 2.4** Singh et al. [1] (Finite mset): A mset $A$ is said to be finite if and only if $A^*$ is finite.

Note that $A$ is finite if and only if $|A| < \infty$

**Definition 2.5** Singh et al. [1]: A domain $X$, is defined as a set of elements from which msets are drawn. We denote the mset space $M(X)$ as the set of all finite msets whose elements are in $X$. i.e

If $X = \{x_1, x_2, \dots, x_n\}$, then $M(X) = \{\{m_1/x_1, m_2/x_2, m_3/x_3, \dots, m_n/x_n\}\}$

$x_i \in X, i = 1,2,3, \dots, n$. Note that $M^* \in M(X)$ for which $C_{M^*}(x) = n \leftrightarrow n = 1$ ([5])

**Definition 2.6** Singh et al. [1] (**mset relations and operations)**:

Let $M, N \in M(X)$. Then

- **i.** **(equality)** $M = N$ iff $C_M(x) = C_N(x)\ \forall x \in X$.
- **ii.** **(submset)** $M \subseteq N$ iff $C_M(x) \leq C_N(x) \forall x \in X$. Note that $\emptyset \subseteq N\ \forall N \in M(X)$
- **iii** **(mset union)** $P = M \uplus N$ iff $C_P(x) = Max\{C_M(x), C_N(x)\}\ \forall x \in X$.
- **iv** **(mset intersection)** $P = M \cap N$ iff $C_P(x) = Min\{C_M(x), C_N(x)\} \forall x \in X$
- **v** **(mset addition)** $P = M \oplus N$ iff $C_P(x) = C_M(x) + C_N(x) \forall x \in X$
- **vi** **(mset Difference**) $P = M \ominus N$ iff $C_P(x) = Max\{C_M(x) - C_N(x), 0\}$l $\forall x \in X$.
- **viii** (**mset arithmetic multiplication)** $P = M \odot N\ iff\ C_P(x) = C_M(x).C_N(x)\ \forall x \in X$.
- **ix** **(mset raisig to arithmetic power)**

  $P = M^n$ iff $C_P(x) = C_{M^n}(x) = \left(C_M(x)\right)^n$.

- **x** **(mset scalar multiplication)** $P = kM$ for $k \in \{1,2, \dots\}$ iff $C_P(x) = kC_M(x)$

all $x \in X$.

Note that the operations $\uplus, and\ \cap$ ara commutative, associative and distributive over each other(see([1,18]). It is also clear from the definitions that for any $A, B \in M(X)$, we have

$(A \uplus B)^n = A^n \uplus B^n$ and $(A \cap B)^n = A^n \cap B^n$ where $n \in N$, the set of nonnegative integers.

**Definition 2.7** Gerish and Sumil **[2] (power mset):**

Let $M\ \in M(X)$ be an mset. The power mset $P(M)$ of $M$ is the mset of all submsets of $M$.
For example given that $M = [x, x, y]$, we

have $P(M) = [\emptyset, \{x\}, \{x\}, \{y\}, [x, x], \{x, y\}, \{x, y\}, [x, x, y]]$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Clearly, we have $P^*(M) = [\emptyset, \{x\}, \{y\}, [x, x], \{x, y\}, [x, x, y]]$ and $P^*(M) \neq P(M^*)$.

**Definition 2.8** Gerish and Sumil [2] **(**M-topological space): Let $M\ \in M(X)$ and $\tau \subseteq P^*(M)$. Then $\tau$ is called an mset topology on $M$ if $\tau$ satisfies the following properties:

1. The mset $M$ and the empty mset $\phi$ are in $\tau$.
2. The mset union of the elements of any members of $\tau$ is in $\tau$.
3. The mset intersection of the elements of any finite subcollection of $\tau$ is in $\tau$.

The ordered pair $(M, \tau)$ is called M-topological space. Each element in $\tau$ is called open mset.

For example, let $X$ be a non empty set and $\emptyset \neq M \in M(X)$. Then $(M, \tau)$ is an

M-topological space where $\tau = P^*(M)$. We denote the class of M-topological spaces by $MTOP$

**Definition 2.9** Grish and Sumil [9] (Submulti space of an M-topology space): Let $(M, \tau)$ be an

M-topological space such that $M \in M(X)$ and $N \subseteq M$. Then the ordered pair $(N, \tau_N)$ such that

$\tau_N = \{U \in M(X): U = N \cap V, V \in \tau\}$ is called submspace of the M- topological space $(M, \tau)$.

**Definition 2.10** Sobhy et al. [12] (Hausdorff M-topological Space**):** Let $(M, \tau)$ be an M-topological space where $M \in M(X)$**.** If for every two simple msets $\{k_1/x_1\}, \{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$, then there exist $G, H \in \tau$ such that $\{k_1/x_1\} \subseteq G, \{k_2/x_2\} \subseteq H$ and $G \cap H \doteq \phi$. Then $(M, \tau)$ is said to be a Hausdorff M- topological space.We denote the class of Hausdorff M-topological spaces by $HMTOP$

**Definition 2.11** Nelson [19] ($M$-Bitopological Space): An $M$-bitopological space is a triple $(M, \tau_1, \tau_2)$ where $M \in M(X)$ and $\tau_1, \tau_2$ are arbitrary M-topologies on $M$. We denote the class of M-bitopological spaces by $MBTOP$

**Definition 2,12:** Let $(M, \tau_1, \tau_2)$ be $M$-Bitopological Space where $M \in M(X)$ and $N \subset M$. Then $(N, \tau_{1N}, \tau_{2N})$ where $\tau_{1N} = \{A = U \cap N, U \in \tau_1\}$ $and$ $\tau_{2N} = \{B = V \cap N, V \in \tau_2\}$ is called a submspace of the $M$ - Bitopological Space.

# 3 Some Results

**Proposition 3.1:** Let $(M, \tau) \in MTOP$ and $N, M \in M(X)$such that $N \subseteq M$. Then the submspace $(N, \tau_N) \in MTOP$.

**Proof:**

Let $(M, \tau) \in MTOP$ and $N, M \in M(X)$ such that $N \subseteq M$

Then we show that $(N, \tau_N) \in MTOP$.

Since by definition, we have $\tau_N = \{U: U = N \cap V, V \in \tau\}$

But $\phi = N \cap \phi$ (since $\phi$ is an open set in $(M, \tau)$) then $\phi$ is also open in $(N, \tau_N)$.

Also, $N = N \cap M$ and $M$ is open in $(M, \tau)$, then $N$ is also open in $(N, \tau_N)$. Thus $\phi, N \in \tau_N$.

Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of open sets in $\tau_N$, for each $\alpha \in I$ there exist open set $\beta_\alpha \in \tau$ such that $A_\alpha = N \cap \beta_\alpha$. This implies $\uplus_{\alpha \in I} A_\alpha = \uplus_{\alpha \in I} (N \cap \beta_\alpha) = N \cap (\uplus_{\alpha \in I} (\beta_\alpha)) \in \tau_N$

(since $\uplus_{\alpha \in I} (\beta_\alpha) \in \tau$). Therefore, the arbitrary union $\bigcup_{\alpha \in I} A_\alpha$ is open in $\tau_N$.

Furthermore, supposed $A_1, \dots, A_n$ are open msets in $\tau_N$. Then there exist $\beta_1, \dots, \beta_n \in \tau$ such that $A_i = N \cap \beta_i$. This imples $\cap_{i \doteq 1}^n A_i = \cap_{i \doteq 1}^n (N \cap \beta_i) = N \cap (\cap_{i \doteq 1}^n \beta_i) \in \tau_N$ (since $\cap_{i \doteq 1}^n \beta_i \in \tau$)

Therefore, $\cap_{i \doteq 1}^n A_i$ is open in $\tau_N$. Hence, $(N, \tau_N)$ is M-topological space.

In particular, $(N, \tau_N) \in MTOP$

**Definition 3.2 (Union):** Let $(M, \tau_1), (N, \tau_2) \in MTOP$ where $M, N \in M(X)$.
The union of these M-topological spaces denoted $(M, \tau) \sqcup (N, \tau_2)$ is given by:

$(M, \tau) \sqcup (N, \tau_2) = (M \uplus N, \tau_1 \Cup \tau_2)$ where $\tau_1 \Cup \tau_2 = \{U \uplus V | U \in \tau_1, V \in \tau_2\}$

**Definition3.3(Intersection):** Let $(M, \tau_1), (N, \tau_2) \in MTOP$ where $M, N \in M(X)$.

The intersection of these M-topological spaces denoted $(M, \tau) \sqcap (N, \tau_2)$ is given by:

$(M, \tau) \sqcap (N, \tau_2) = (M \cap N, \tau_1 \Cap \tau_2)$ where $\tau_1 \Cap \tau_2 = \{U \cap V | U \in \tau_1, V \in \tau_2\}$

**Definition3.4 (arithmetic multiplication):** Let $(M, \tau_1), (N, \tau_2) \in MTOP$ where $M, N \in M(X)$

The arithmetic multiplication of these M-topological spaces denoted $(M, \tau) \widetilde{\odot} (N, \tau_2)$ is given by: $(M, \tau) \widetilde{\odot} (N, \tau_2) = (M \odot N, \tau_1 \odot \tau_2)$ where $\tau_1 \odot \tau_2 = \{U \odot V | U \in \tau_1, V \in \tau_2\}$

**Definition3.5 (mset multiplication):** Let $(M, \tau) \in MTOP$ where $M, N \in M(X)$

The mset multiplication $N \odot (M, \tau)$ is defined by $N \odot (M, \tau) = (N \odot M, N \odot \tau)$

where $N \odot \tau = \{N \odot U | U \in \tau\}$

**Definition 3.6:** Let $(M, \tau) \in MTOP$ where $M \in M(X)$. The scalar multiplication

$k(M, \tau)$ is defined by $k(M, \tau) = (kM, k\tau)$ where $k\tau = \{kU | U \in \tau\}$ and $k \in \{1,2, ...\}$.

**Definition 3.7:** Let $(M, \tau) \in MTOP$ where $M \in M(X)$. Raising $(M, \tau)$ to arithmetic

power $n$ denoted $(M, \tau)^n$ is defined by $(M, \tau)^n = (M^n, \tau^n)$

where $\tau^n = \{U^n | U \in \tau\}$, $n \in \{1,2, ...\}$.

**Proposition 3.8:** Let $M, N \in M(X)$. Then

- **i.** $(M, \tau_1) \wedge (N, \tau_2) \in MTOP \Longrightarrow (M, \tau_1) \sqcup (N, \tau_2) \in MTOP$
- **ii.** $(M, \tau_1) \wedge (N, \tau_2) \in MTOP \Longrightarrow (M, \tau_1) \sqcap (N, \tau_2) \in MTOP$
- iii. $(M, \tau) \in MTOP \Longrightarrow N \odot (M, \tau) \in MTOP$
- iv. $(M, \tau) \in MTOP \Longrightarrow (M, \tau)^n \in MTOP$
- v. $(M, \tau) \in MTOP \Longrightarrow k(M, \tau) \in MTOP$

**Proof:**

i. **Now** $(M, \tau_1) \sqcup (N, \tau_2) = (M \uplus N, \tau_1 \Cup \tau_2)$ where $\tau_1 \Cup \tau_2 = \{U \uplus V | U \in \tau_1, V \in \tau_2\}$

Note that $U \subseteq M \wedge V \subseteq N \Longrightarrow U \uplus V \subseteq M \uplus N$.

Since $\emptyset, M \in \tau_1$ and $\emptyset, N \in \tau_2$, we have $\emptyset \uplus \emptyset = \emptyset, M \uplus N \in \tau_1 \Cup \tau_2$ (1)

Let $\{U_\alpha \uplus V_\beta\}_{\alpha,\beta \in I}$ be a collection of open sets in $\tau_1 \Cup \tau_2$, for each $\alpha, \beta \in I$

Now $\uplus \{U_\alpha \uplus V_\beta\} = (\uplus U_\alpha) \uplus (\uplus V_\beta)$.

Since $\uplus U_\alpha \in \tau_1$ and $\uplus V_\beta \in \tau_2$ (by hypothesis)

We have $(\uplus U_\alpha) \uplus (\uplus V_\beta) \in \tau_1 \Cup \tau_2$(by definition)

In particular, $\uplus \left\{U_\alpha \uplus V_\beta\right\} \in \tau_1 \Cup \tau_2$ (2)

Let $\{U_1 \uplus V_1, U_2 \uplus V_2, \dots, U_n \uplus V_n\}$ be any finite collections in $\tau_1 \Cup \tau_2$

We have $\Cap_{i=1}^{n} (U_i \uplus V_i) = (\Cap_{i=1}^{n} U_i) \uplus (\Cap_{i=1}^{n} V_i)$.

Since $\Cap_{i=1}^{n} U_i \in \tau_1$ and $\Cap_{i=1}^{n} V_i \in \tau_2$(by hypothesis), then we have

$$\Cap_{i=1}^{n} (U_i \uplus V_i) = (\Cap_{i=1}^{n} U_i) \uplus (\Cap_{i=1}^{n} V_i) \in \tau_1 \Cup \tau_2 \quad (3)$$

It's clear that $(M, \tau_1) \sqcup (N, \tau_2) \in MTOP$ (from 1-3)

ii. Now,$(M, \tau) \sqcap (N, \tau_2) = (M \Cap N, \tau_1 \Cap \tau_2)$ where $\tau_1 \Cap \tau_2 = \{U \Cap V | U \in \tau_1, V \in \tau_2\}$

Note that $U \subseteq M \wedge V \subseteq N \Longrightarrow U \Cap V \subseteq M \Cap N$

Since $\emptyset, M \in \tau_1$ and $\emptyset, N \in \tau_2$, we have $\emptyset \Cap \emptyset = \emptyset, M \Cap N \in \tau_1 \Cap \tau_2$ (4)

Let $\left\{U_\alpha \Cap V_\beta\right\}_{\alpha,\beta \in I}$ be a collection of open msets in $\tau_1 \Cap \tau_2$, for each $\alpha, \beta \in I$

**Now** $\Cap \left\{U_\alpha \Cap V_\beta\right\} = (\Cap U_\alpha) \Cap \left(\Cap V_\beta\right)$.

Since $\Cap U_\alpha \in \tau_1$ and $\Cap V_\beta \in \tau_2$ (by hypothesis), We have $(\Cap U_\alpha) \Cap \left(\Cap V_\beta\right) \in \tau_1 \Cap \tau_2$

(by definition). In particular, $\Cap \left\{U_\alpha \Cap V_\beta\right\} \in \tau_1 \Cap \tau_2$ (5)

Let $\{U_1 \Cap V_1, U_2 \Cap V_2, \dots, U_n \Cap V_n\}$ be any finite collections in $\tau_1 \Cap \tau 2$

We have $\uplus_{i=1}^{n} (U_i \Cap V_i) = (\uplus_{i=1}^{n} U_i) \Cap (\uplus_{i=1}^{n} V_i)$.

Since $\uplus_{i=1}^{n} U_i \in \tau_1$ and $\uplus_{i=1}^{n} V_i \in \tau_2$(by hypothesis), then we have

$$\uplus_{i=1}^{n} (U_i \Cap V_i) = (\uplus_{i=1}^{n} U_i) \Cap (\uplus_{i=1}^{n} V_i) \in \tau_1 \Cap \tau_2 \quad (6)$$

It's clear that $(M, \tau_1) \sqcap (N, \tau_2) \in MTOP$ (from 4-6)

iii. Now $N \odot (M, \tau) = (N \odot M, N \odot \tau)$ where $N \odot \tau = \{N \odot U | U \in \tau\}$(by definition)

Note that $U \subseteq M \Longrightarrow N \odot U \subseteq N \odot M$.

Since $\emptyset, M \in \tau$, we have $N \odot \emptyset = \emptyset \in N \odot \tau$ and $N \odot M \in N \odot \tau$ (7)

Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of open sets in $N \odot \tau$, for each $\alpha \in I$ there exist open set $\beta_\alpha \in \tau$ such that $A_\alpha \doteqdot N \odot \beta_\alpha$. This implies:

$$\uplus_{\alpha \in I} A_\alpha \doteqdot \uplus_{\alpha \in I} (N \odot \beta_\alpha) \doteqdot N \odot (\uplus_{\alpha \in I} (\beta_\alpha)) \in N \odot \tau \quad (8)$$

(since $\uplus_{\alpha \in I} (\beta_\alpha) \in \tau$ by hypothesis)

Therefore, the arbitrary union $\bigcup_{\alpha \in I} A_\alpha$ is open in $N \odot \tau$.

Furthermore, Supposed $A_1, \ldots, A_n$ are open msets in $N \odot \tau$

Then there exist $\beta_1, \ldots, \beta_n \in \tau$ such that $A_i \doteqdot N \odot \beta_i$. This imples

$\Cap_{i \doteqdot 1}^{n} A_i \doteqdot \Cap_{i \doteqdot 1}^{n} (N \odot \beta_i) \doteqdot N \odot (\Cap_{i \doteqdot 1}^{n} \beta_i) \in N \odot \tau$ (since $\Cap_{i \doteqdot 1}^{n} \beta_i \in \tau$) by hypothesis)

Therefore, $\Cap_{i \doteqdot 1}^{n} A_i \in N \odot \tau$ (9)

Hence, $N \odot (M, \tau) \in MTOP$ (from (7-9)

iv. Now $(M, \tau)^n = (M^n, \tau^n)$ where $\tau^n = \{U^n | U \in \tau\}, n \in \{0,1,2, \ldots\}$ (by definition)

Thus, for $n = 0$, we have $(M, \tau)^0 \in MTOP$ ([10])

Note that $U \sqsubseteq M \Longrightarrow U^n \sqsubseteq M^n$

For $n > 0$ we have $\phi = \phi^n, M^n \in \tau^n$ (Since $\phi, M \in \tau$ (by hypothesis)) (10)

Let $\{U_\alpha^n\}_{\alpha \underline{\in} I}$ be a collection of msets in $\tau^n$

Now $\uplus U_\alpha^n \doteqdot (\uplus U_\alpha)^n$ ([9]). But $\uplus U_\alpha \in \tau$(by hypothesis)

Thus $(\uplus U_\alpha)^n \in \tau^n$ .In particular, $\uplus U_\alpha^n \in \tau^n$ (11)

supposed $\{U_1^n, U_2^n \ldots, U_r^n\}$ be a finite collection of msets in $\tau^n$ such that $U_i \in \tau$

Now $\Cap_{i \doteqdot 1}^{r} U_i^n \doteqdot (\Cap_{i \doteqdot 1}^{r} U_i)^n$ ([9]). But $\Cap_{i \doteqdot 1}^{r} U_i \underline{\in} \tau$ (by hypothesis).

Thus, $(\Cap_{i \doteqdot 1}^{r} U_i)^n \in \tau^n$.

In particular, $\Cap_{i \doteqdot 1}^{r} U_i^n \in \tau^n$ (12)

Hence, $(M, \tau)^n \in MTOP$ (from (10-12) above)

v. Note that $U \sqsubseteq M \Longrightarrow kU \sqsubseteq kM$

Now $k(M, \tau) = (kM, k\tau)$ where $k\tau = \{kU | U \in \tau\}$ and $k \in \{1,2, \ldots\}$(by definition)

Since $(M, \tau) \in MTOP$ then $\phi, M \in \tau$ and $k\phi = \phi, kM \in k\tau$ (by hypothesis) (13)

Let $\{kA_\alpha\}_{\alpha \in I}$ be a collection of msets for each $\alpha \in I$ and $A_\alpha \in \tau$

Now $\uplus kA_\alpha = k\uplus A_\alpha$ .

But $\uplus A_\alpha \in \tau$ (by hypothesis)

Thus, $k\uplus A_\alpha \in k\tau$

In particular, $\uplus kA_\alpha \in k\tau$ (14)

Let $\{kA_1, kA_2, \ldots, kA_n\}$ be a finite collection of msets in $k\tau$ such that $A_i \in \tau$ .

Now $\Cap_{i \doteqdot 1}^{n} kA_i = k(\Cap_{i \doteqdot 1}^{n} A_i)$

But $\Cap_{i \doteqdot 1}^{n} A_i \in \tau$ (by hypothesis)

Thus, $k(\cap_{i\doteq 1}^{n} A_i) \in k\tau$

Thus, $\cap_{i\doteq 1}^{n} kA_i \in k\tau$ (15)

In particular, $k(M,\tau) \in MTOP$(fron (13-15) above)

Proposition 3.9: Let $(M,\tau) \in HMTOP$ and $N, M \in M(X)$ such that $N \subseteq M$. Then the submspace $(N,\tau_N) \in HMTOP$

Proof:

Clearly, $(N,\tau_N) \in MTOP$ (by proposition 3.1)

Let $\{k_1/x\}$ and $\{k_2/y\}$ be two simple msets such that $\{k_1/x\}, \{k_2/y\} \subseteq N$ and $x \neq y$

Thus, $\{k_1/x\}, \{k_2/y\} \subseteq M$ (since $N \subseteq M$)

We have $U, V \in \tau$ such that

$\{k_1/x\} \subseteq U$, $\{k_2/y\} \subseteq V$and $U \cap V = \phi$ (by hypothesis and definition)

Thus, we have $A, B \in \tau_N$ such that $A = N \cap U$ and $B = N \cap V$

But $A \cap B = (N \cap U) \cap (N \cap V) = N \cap (U \cap V) = N \cap \emptyset = \emptyset$

Since $\{k_1/x\}, \{k_2/y\} \subseteq N$ and $\{k_1/x\} \subseteq U$, $\{k_2/y\} \subseteq V$, we have $\{k_1/x\} \subseteq N \cap U = A$ and $\{k_2/y\} \subseteq N \cap V = B$ such that $A \cap B = \emptyset$

So that $(N,\tau_N) \in HMTOP$

**Proposition 3.10:** Let $(M,\tau) \in HMTOP$ where $M \in M(X)$. Then

i. $(M,\tau)^n \in HMTOP$
ii. $k(M,\tau) \in HMTOP$

**Proof:**

i. Given that $(M,\tau) \in HMTOP$. It's clear that $(M,\tau) \in MTOP$ (by definition)

Thus, $(M,\tau)^n = (M^n,\tau^n) \in MTOP$ (by proposition 3.8)

Let $\{k_1^n/x\}$ and $\{k_2^n/y\}$ be two simple msets such that $\{k_1^n/x\}, \{k_2^n/y\} \subseteq M^n$ and $x \neq y$ where $k_1, k_2 > 0$ and $\{k_1/x\}, \{k_2/y\} \subseteq M$.

Thus we have $U, V \in \tau$ such that

$\{k_1/x\} \subseteq U$, $\{k_2/y\} \subseteq V$ and $U \cap V = \emptyset$ (by hypothesis and definition) (16)

But But $\{k_1/x\} \subseteq U \Longrightarrow \{k_1^n/x\} \subseteq U^n$, (17)

$\{k_2/y\} \subseteq V \Longrightarrow \{k_2^n/y\} \subseteq V^n$ (18)

and

$U^n \cap V^n = (U \cap V)^n = \emptyset^n = \emptyset$ ([9]) (19)

Since $U, V \in \tau$, we have $U^n, V^n \in \tau^n$ (by definition)

In particular $(M^n, \tau^n) \in HMTOP$ (from (16-19)

ii. Given that $(M, \tau) \in HMTOP$. It's clear that $(M, \tau) \in MTOP$ (by definition)

Thus, $k(M, \tau) = (kM, k\tau) \in MTOP$ (by proposition 3.8)

Let $\{kk_1/x\}$ and $\{kk_2/y\}$ be two simple msets such that $\{kk_1/x\}, \{kk_2/y\} \sqsubseteq kM$ and $x \neq y$ where $\{k_1/x\}, \{k_2/y\} \sqsubseteq M$.

Thus we have $U, V \in \tau$ such that $\{k_1/x\} \sqsubseteq U$, $\{k_2/y\} \sqsubseteq V$ and $U \cap V = \emptyset$ ( definition), (20)

$$\text{But } \{k_1/x\} \sqsubseteq U \Longrightarrow \{kk_1/x\} \sqsubseteq kU, \tag{21}$$

$$\{k_2/y\} \sqsubseteq V \Longrightarrow \{kk_2/y\} \sqsubseteq kV \tag{22}$$

$$kU \Cap kV \doteqdot k(U \Cap V) = k\emptyset = \emptyset([9]) \tag{23}$$

Since $U, V \in \tau$, we have $kU, kV \in k\tau$(by definition)

In particular $(kM, k\tau)$ is Hausdorff (from 20-23)

Thus, $k(M, \tau) \in HMTOP$

Proposition 3.11: Let $(M, \tau_1, \tau_2) \in MBTOP$ where $M, N \in M(X)$ such that $N \sqsubset M$ . Then $(N, \tau_{1N}, \tau_{2N}) \in MBTOP$.

Proof:

Since $\tau_{1N} = \{A = U \Cap N, U \in \tau_1\}, \tau_{2N} = \{B = V \Cap N, V \in \tau_2\}$ (by definition)

It's clear that $(N, \tau_{1N}), (N, \tau_{2N}) \in MTOP$ (Proposition 3.1)

In particular, $(N, \tau_{1N}, \tau_{2N}) \in MBTOP$ (by definition)

**Definition 3.12:** Let $(M, \tau_1, \tau_2), (M_1, \tau_{11}, \tau_{12}), (M_2, \tau_{21}, \tau_{22}) \in MBTOP$

where $M_1, M_2, M, N \in M(X)$. We define union(⊔), intersection(⊓), mset multiplication and raising to arithmetic power as follows respectively:

$(M_1, \tau_{11}, \tau_{12}) \sqcup (M_2, \tau_{21}, \tau_{22}) = \left(\left(\uplus_{j=1}^{2} M_j\right), \tau_{1i} \Cup \tau_{2i}, \right), i = 1,2$ where

$\tau_{1i} \Cup \tau_{2i} = \{U_{1i} \uplus V_{2i} | U_{1i} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$,,

$(M_1, \tau_{11}, \tau_{12}) \sqcap (M_2, \tau_{21}, \tau_{22}) = \left(\left(\Cap_{j=1}^{2} M_j\right), \tau_{1i} \Cap \tau_{2i}, \right), i = 1,2$ where

$\tau_{1i} \Cap \tau_{2i} = \{U_{1i} \cap V_{2i} | U_{1i} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$,

$N \odot (M, \tau_1, \tau_2) = (N \odot M, N \odot \tau_1, N \odot \tau_2)$ where $N \odot \tau_1 = \{N \odot U | U \in \tau_1\}$ and

$N \odot \tau_2 = \{N \odot V | V \in \tau_2\}$.

$(M, \tau_1, \tau_2)^n = (M^n, \tau_1^n, \tau_2^n)$ where $\tau_1^n = \{U^n | U \in \tau_1\}$ and $\tau_2^n = \{V^n | V \in \tau_2\}$

$k(M, \tau_1, \tau_2) = (kM, k\tau_1, k\tau_2)$ where $k\tau_1 = \{kU | U \in \tau_1\}$ and $k\tau_2 = \{kV | V \in \tau_2\}$

**Proposition 3.13:** Let $(M, \tau_1, \tau_2), (M_1, \tau_{11}, \tau_{12}), (M_2, \tau_{21}, \tau_{22}) \in MBTOP$

where $M_1, M_2, M, N \in M(X)$ . Then

- **i.** $(M_1, \tau_{11}, \tau_{12}) \sqcup (M_2, \tau_{21}, \tau_{22}) \in MBTOP$
- **ii.** $(M_1, \tau_{11}, \tau_{12}) \sqcap (M_2, \tau_{21}, \tau_{22}) \in MBTOP$
- iii. $N \odot (M, \tau_1, \tau_2) \in MBTOP$
- iv. $(M, \tau_1, \tau_2)^n \in MBTOP$
- v. $k(M, \tau_1, \tau_2) \in MBTOP$
- vi. $(M, \tau_1 \cap \tau_2) \in MTOP$

Proof:

i. $(M_1, \tau_{11}, \tau_{12}) \sqcup (M_2, \tau_{21}, \tau_{22}) = \left(\left(\uplus_{j=1}^{2} M_j\right), \tau_{1i} \Cup \tau_{2i}, \right), i = 1,2$ where

$\tau_{1i} \Cup \tau_{2i} = \{U_{1i} \uplus V_{2i} | U_{1i} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$ (by definition)

Now $\left(\left(\uplus_{j=1}^{2} M_j\right), \tau_{1i} \Cup \tau_{2i}, \right) \in MTOP \; \forall i$ (Proposition 3.8)

In particular, $(M_1, \tau_{11}, \tau_{12}) \sqcup (M_2, \tau_{21}, \tau_{22}) \in MBTOP$(by definition)

ii. $(M_1, \tau_{11}, \tau_{12}) \sqcap (M_2, \tau_{21}, \tau_{22}) = \left(\left(\cap_{j=1}^{2} M_j\right), \tau_{1i} \Cap \tau_{2i}, \right), i = 1,2$ where

$\tau_{1i} \Cap \tau_{2i} = \{U_{1i} \cap V_{2i} | U_{1i} \in \tau_{1i}, V_{2i} \in \tau_{2i}\}$(by definition)

Now $\left(\left(\cap_{j=1}^{2} M_j\right), \tau_{1i} \Cap \tau_{2i}, \right) \in MTOP \; \forall i$ (Proposition 3.8)

In particular, $(M_1, \tau_{11}, \tau_{12}) \sqcap (M_2, \tau_{21}, \tau_{22}) \in MBTOP$(by definition)

iii. $N \odot (M, \tau_1, \tau_2) = (N \odot M, N \odot \tau_1, N \odot \tau_2)$ where $N \odot \tau_1 = \{N \odot U | U \in \tau_1\}$

and $N \odot \tau_2 = \{N \odot V | V \in \tau_2\}$(by definition)

Now $(N \odot M, N \odot \tau_1), (N \odot M, N \odot \tau_2) \in MTOP$(Proposition 3.8)

In particular, $(N \odot M, N \odot \tau_1, N \odot \tau_2) \in MBTOP$(by definition)

iv. $(M, \tau_1, \tau_2)^n = (M^n, \tau_1^n, \tau_2^n)$ where $\tau_1^n = \{U^n | U \in \tau_1\}$ and $\tau_2^n = \{V^n | V \in \tau_2\}$

(by definition). Now, $(M^n, \tau_1^n), (M^n, \tau_2^n) \in MTOP$(Proposition 3.8)

In particular, $(M, \tau_1, \tau_2)^n = (M^n, \tau_1^n, \tau_2^n) \in MBTOP$(by definition)

v. $k(M, \tau_1, \tau_2) = (kM, k\tau_1, k\tau_2)$ where $k\tau_1 = \{kU | U \in \tau_1\}$ and $k\tau_2 = \{kV | V \in \tau_2\}$

(by definition). Now $(kM, k\tau_1), (kM, k\tau_2) \in MTOP$(Proposition 3.8)

In particular, $k(M, \tau_1, \tau_2) = (kM, k\tau_1, k\tau_2) \in MBTOP$(by definition)

vi. Given that $(M, \tau_1, \tau_2) \in MBTOP$, we show that $(M, \tau_1 \cap \tau_2) \in MTOP$

Since $\tau_1$ and $\tau_2$ are two topologies defined on $M$. Then $\phi, M \in \tau_1$ and $\phi, M \in \tau_2$

(by definition). Thus, $\phi, M \in \tau_1 \cap \tau_2$ (24)

Let $\{A_\alpha \in \tau_1 \cap \tau_2\}$ be an arbitrary collection Then we show that

$\uplus A_{\alpha \in I} \in \tau_1 \cap \tau_2$ . Now $A_\alpha \in \tau_1 \cap \tau_2 \Longrightarrow A_\alpha \in \tau_1$ and $A_\alpha \in \tau_2$

But $A_\alpha \in \tau_1 \Longrightarrow \uplus A_{\alpha \in I} \in \tau_1$and $A_\alpha \in \tau_2 \Longrightarrow \uplus A_{\alpha \in I} \in \tau_2$ (by hypothesis and definition).

Thus, $A_\alpha \in \tau_1 \cap \tau_2 \Longrightarrow \uplus A_{\alpha \in I} \in \tau_1 \cap \tau_2$ (25)

Let $\{A_1, A_2, \dots, A_n\}$ be a finite collections such that $A_i \in \tau_1 \cap \tau_2$

Now $A_i \in \tau_1 \cap \tau_2 \Longrightarrow A_i \in \tau_1$ and $A_i \in \tau_2$. But $A_i \in \tau_1 \Longrightarrow \Cap_{i=1}^{n} A_i \in \tau_1$ and

$A_i \in \tau_2 \Longrightarrow \Cap_{i=1}^{n} A_i \in \tau_2$ (by hypothesis and definition)

In particular, $A_i \in \tau_1 \cap \tau_2 \Longrightarrow \Cap_{i=1}^{n} A_i \in \tau_1 \cap \tau_2$ (26)

Thus $(M, \tau_1 \cap \tau_2) \in MTOP$ (from 24-26 above)

**Definition 3.14:** Let $(M, \tau_1, \tau_2) \in MBTOP$. Then $(M, \tau_1, \tau_2)$

Is a Hausdorff M-bitopological space if and only if for any

simple mset $\{k_1/x\}, \{k_2/y\} \subseteq M$ with $x \neq y$, there exist $U \in \tau_1$ and $V \in \tau_2$

such that $\{k_1/x\} \subseteq U$, $\{k_2/y\} \subseteq V$ and $U \cap V = \phi$,

**Example 3.15:**

Let $M = \{3/a\,, 4/b\,, 2/c\,, 1/d\}$ be an mset, $\tau_1 = \{X, \phi, \{3/a\}, \{2/b\}, \{3/a\,, 2/b\}\}$ and

$\tau_2 = \{X, \phi, \{2/a\}, \{2/c\}, \{2/a\,, 2/c\}\}$ be two $M$-topologies. Then, $(M, \tau_1, \tau_2)$ is Hausdorff mset bitopological space. We denote the class of Hausdorff mset bitopological space by $HMBTOP$

**Proposition 3.16:** Let $(M, \tau_1, \tau_2) \in HMBTOP$ and $N \in M(X)$

such that $N \subseteq M$. Then $(N, \tau_{1N}, \tau_{2N}) \in HMBTOP$

**Proof:**

Let $(M, \tau_1, \tau_2) \in HMBTOP$ and $(N, \tau_{1N}, \tau_{2N})$ be its submspace. The we show that $(N, \tau_{1N}, \tau_{2N})$ is Hausdorff M-bitopological space. Clearly

$(N, \tau_{1N}, \tau_{2N}) \in MBTOP \in$ (by definition and proposition 3.11).

Let $\{k_1/x\}, \{k_2/y\} \subseteq N$ such that $x \neq y$. Since $N \subseteq M$, then $\{k_1/x\}, \{k_2/y\} \subseteq M$.

Since $(M, \tau_1, \tau_2) \in HMBTOP$(by hypothesis), we have $A \in \tau_1$ and $B \in \tau_2$ such that

$\{k_1/x\} \subseteq A$, $\{k_2/y\} \subseteq B$ and $A \Cap B = \phi$

But $A \in \tau_1 \Longrightarrow N \Cap A \in \tau_{1N}$ and $B \in \tau_2 \Longrightarrow N \Cap B \in \tau_{2N}$. In particular, $\{k_1/x\} \subseteq N \Cap A$ and

$\{k_2/y\} \subseteq N \Cap B$. But $(N \Cap A) \Cap (N \Cap B) = N \Cap (A \Cap B) = N \Cap \emptyset = \emptyset$

Hence, $(N, \tau_{1N}, \tau_{2N}) \in HMBTOP$.

**Proposition 3.17:** Let $(M, \tau_1, \tau_2) \in HMBTOP$. Then

i. $(M, \tau_1, \tau_2)^n \in HMBTOP$ for $n \in \{1,2, \dots\}$
ii. $k(M, \tau_1, \tau_2) \in HMBTOP$ for $n \in \{1,2, \dots\}$

Proof:

i. Clearly, $(M, \tau_1, \tau_2) \in HMBTOP \Longrightarrow (M, \tau_1, \tau_2) \in MBTOP$ (by definition)

Thus, $(M, \tau_1, \tau_2)^n \in MBTOP$ (Proposition 3.13)

We show that $(M, \tau_1, \tau_2)^n$ is Hausdorff

Now let $\{k_1^n/x\}, \{k_2^n/y\} \subseteq M^n$ such that $\{k_1/x\}, \{k_2/y\} \subseteq M$ and $x \neq y$

Thus, we have $U \in \tau_1$ and $V \in \tau_2$ such that $\{k_1/x\} \subseteq U$, $\{k_2/y\} \subseteq V$ and $U \cap V = \emptyset$

**(**by hypothesis and definition)

But $\{k_1/x\} \subseteq U \Longrightarrow \{k_1^n/x\} \subseteq U^n$ and $\{k_2/y\} \subseteq V \Longrightarrow \{k_2^n/y\} \subseteq V^n$

Note that $U \in \tau_1 \Longrightarrow U^n \in \tau_1^n$ and $V \in \tau_2 \Longrightarrow V^n \in \tau_2^n$(by definition) .

Thus, $U^n \cap V^n = (U \cap V)^n = \emptyset^n = \emptyset$ and $(M, \tau_1, \tau_2)^n = (M^n, \tau_1^n, \tau_2^n)$ is Hausdorff.

In particular, $(M, \tau_1, \tau_2)^n \in HMBTOP$.

ii. Clearly, $k(M, \tau_1, \tau_2) \in HMBTOP \Longrightarrow k(M, \tau_1, \tau_2) \in MBTOP$ (by definition)

Thus, $k(M, \tau_1, \tau_2) \in MBTOP$(Proposition 3.13)

We show that $k(M, \tau_1, \tau_2)$ is Hausdorf. Since $k(M, \tau_1, \tau_2) = (kM, k\tau_1, k\tau_2)$ where

$k\tau_1 = \{kU | U \in \tau_1\}$ and $k\tau_2 = \{kV | V \in \tau_2\}$, $k \in \{1,2, \dots\}$ (by definition),

Let $\{k\, k_1/x\}, \{k\, k_2/y\} \subseteq kM$ such that $\{k_1/x\}, \{k_2/y\} \subseteq M$ and $x \neq y$.

Thus, we have $U \in \tau_1$ and $V \in \tau_2$ such that $\{k_1/x\} \subseteq U$, $\{k_2/y\} \subseteq V$ and $U \cap V = \emptyset$

**(**by hypothesis and definition).

But $\{k_1/x\} \subseteq U \Longrightarrow \{k\, k_1/x\} \subseteq kU$ and $\{k_2/y\} \subseteq V \Longrightarrow \{k\, k_2/y\} \subseteq kV$.

Note that $U \in \tau_1 \Longrightarrow kU \in k\tau_1$ and $V \in \tau_2 \Longrightarrow kV \in k\tau_2$(by definition) .

$kU \cap kV = k(U \cap V) = k\emptyset = \emptyset$ and $k(M, \tau_1, \tau_2)$ is a Hausdorff

In particular, $k(M, \tau_1, \tau_2) \in HMBTOP$

# 4 Conclusion

In this article certain operations has been introduced and studied on the various classes of mset topological spaces to include submspace, union, intersection, multiplication, raising to arithmetic power and scalar multiplication. The study of these operations reveals that these operations are closed on the class of M-topological spaces and M-bitopological spaces. However, only the submspace, raising to arithmetic power and scalar multiplication operations are closed on the classes of Hausdorff M-topological space and Hausdorff M-bitopological spaces.

## Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

## Competing Interests

Authors have declared that no competing interests exist.

## References

[1] Singh D, Ibrahim AM, Yohanna T, Singh JN. An overview of the applications of multisets. Novi Sad Journal of Mathematics. 2007;37(2):73-92.

[2] Gerish KP, Sumil Jacob John. Mset Topologies induced by mset relations, Information Sciences. 2012;188:298-313.

[3] Knuth DE. The art of computer programming, Vol. 2 (semi numerical Algorithms), Second Edition, Addison-Wesley, Reading, Massachusetts. 1981;2.

[4] Robert E. Yager, On the Theory of Bags. International journal of general systems; 1986.

[5] Blizard, Wayne D. Mset theory, Notre Dame Journal of Formal Logic 1989a;30:f36-66.

[6] Raafat M, Abd El-Latif AM. On multi weak structures. International Journal of Advances in Mathematics. 2010;2017(6):42-50.

[7] Parker – Rhodes AF. The Theory of Indistinguishables, D. Reidel, Dordrecht; 1981.

[8] Petrovsky AB. Structuring Techniques in Multiset Spaces. Springer. 1997;174-1.

[9] Grish KP, Sumil Jacob John. On mset topologies, Theory and Applications o Mathematics & Computer Science 2012;2:37-52.

[10] Mahanta J, Das D. Semi compactness in multiset topology. 2014;arXiv:1403.5642v2.

[11] Mahanta J, Das D. Boundary and exterior of a multisettopology. Math.GM. 2015ar;Xiv:1501.07193,v1.

[12] Sobhy El-Sheikh, Rajab Omar, Mahmoud Raafat. Separation axioms on mset topological space. Journal of New Theory. 2015;7:11-21.

[13] Hasan Dadas. Advanced decision-making techniques with generalized closed sets in neutrosophic soft bitopological spaces. Journal of Mathematical Structures and Applications. 2024;10(1):8-16.

[14] Kelly JC. Bitopological spaces. London Math. Soc. Proc. 1963;3:71-89.

[15] Topology, 2nd edition book, by James R. Munkres; 2000.

[16] Gambo, Tella. A new look at groupoid and cancellation law, International Journal of Software & Hardware Research in Engineering. 2022;10.

[17] Jena SP, Gosh SK, Tripathy BK. On the theory of bags and lists, Information Sciences. 2001;132:241-254.

[18] Wildberger NJ. A new look at multiset, School of mathematics, UNSW Sydney 2052 Australia; 2003.

[19] Nelson Elaine D. Separation axioms in Topology. Graduate Student Theses, Dissertations, &Professional papers. 1966;6117.



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