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On Some Stochastic Ordering Comparisons for Renewal Processes

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

After a brief introduction to ordinary renewal process, we have defined compound renewal process generated by an ordinary renewal process followed by the distributional properties of the corresponding random variables. A few results are obtained by comparing independent renewal processes with respect to several stochastic orderings between the generating inter-arrival time random variables, like, stochastic order, hazard rate order, likelihood ratio order and variability order, as well as some ageing classes of the generating random variables. Some numerical illustrations are given. The results obtained here appear to be new.

Keywords: Models theoretical; Renewal process; compound renewal process; stochastic ordering; variability ordering; ageing classes.

2010 Mathematics Subject Classification: 60E15; 60K05; 60K10.

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1 Introduction

Ordinary renewal process (ORP) is a counting process which was introduced as a generalization of Poisson process. ORP is applied in various areas, such as replacement and maintenance, reliability, actuarial mathematics, manpower studies, demography, queuing theory, inventory control and so on. In reliability language, the system or a component failure times, since when the observation started, generate a renewal process provided the replacement time or the repair time is negligible in comparison to the inter-failure time scale, and renewal refers to either the replacement of the failed component with a brand new one or the case where it is repaired in a negligibly short period of time to an almost brand new condition (as good as before).

In this article, we look at the comparison of two independent ORPs and compound renewal processes (CRPs) by means of stochastic ordering on the basis of the generating renewal distribution and distribution function (df) of the independent and identically distributed (iid) summands. We have referred the Book "Stochastic processes" by Ross [1] for renewal process and these have been studied extensively. For definitions and details on various stochastic orders, we refer to Shaked and Shanthikumar [2], Nair, Sankaran and Balakrishnan [3], Ravi and Prathibha [4], Mitov and Omey [5], and other references [6, 7, 8, 9, 10, 11, 12]. Definitions used in this article are given in Appendix for ease of reference. We answer the following questions in this article: What are the distributional properties of the rvs associated with these processes? How the renewal function and renewal density behaves for various generating renewal distribution? What are the various stochastic ordering relations while comparing two independent ORPs, and an ORP with a homogeneous Poisson process (HPP)? Whether more arrivals (failures) are occurring for an ORP by a given time t when comparing to another independent ORP? What are the implications in stochastic comparison of two independent ORPs with respect to different ageing classes of life distributions? The article is sequenced as follows: Section 1.1 is a brief introduction to the distributional properties of the corresponding random variables. Section 2 contains the main results and Section 3 has some examples and numerical illustrations. The results obtained here appears to be new.

1.1 Preliminaries

Definition 1.1. A generalized counting process, denoted as $\{\, N(t),\ t \geq 0\}$, is defined as $N(t) = \max\left\{n \geq 0 : S_n = \sum_{i=1}^{n} X_i \leq t\right\}$, where $\{X_i, i = 1, 2, \ldots\}$ is a sequence of independent but not necessarily identically distributed non negative continuous random variables (rvs). The events for the counting process $\{N(t), t \geq 0\}$ are occurring at times $S_1, S_2, \ldots$ while the process was started observing at time 0. Therefore, $X_1 = S_1, X_i = S_i - S_{i-1}$; for $i = 2, 3, \ldots$ can be interpreted as the i-th inter-event time, also known as inter-failure time in reliability literature, whereas the rv $N(t)$ denotes the number of events occurred in the time interval $[0, t]$ for some given time $t, t \geq 0$.

Definition 1.2. If we consider the simpler case of the distribution of the first event/ failure/ arrival time X_1 being identical to those of the inter-event times $X_2, X_3, \ldots$ in Definition (1.1), then the resulting counting process is said to be an ORP, denoted as $\{N_F(t), t \geq 0\}$, generated by the baseline distribution function (df) F, where $X_i \sim F$ for $i = 1, 2, \ldots$.

Remark 1.1. As a special case of ORP as defined above (1.2), when the inter-arrival times are exponentially distributed with constant failure rate parameter $\lambda > 0$, or in other words average time to failure is $\frac{1}{\lambda}$, the resulting counting process is known as homogeneous Poisson process (HPP).

Probability Mass Function

For the ORP $\{N_F(t), t \geq 0\}$ as defined in (1.2), $S_n = X_1+X_2+\ldots+X_n \sim F_n$, where F_n is the n-fold convolution of F with itself, as $X_i, i = 1, 2, \ldots, n$ are iid rvs having df F. By definition, n-th event occurs prior to or at time t iff the number of events occurring by time t is at least n, that is $S_n \leq t \Leftrightarrow N_F(t) \geq n$, therefore

$$\mathrm{P}(N_F(t) \geq n) = \mathrm{P}(S_n \leq t) = F_n(t), \tag{1.1}$$

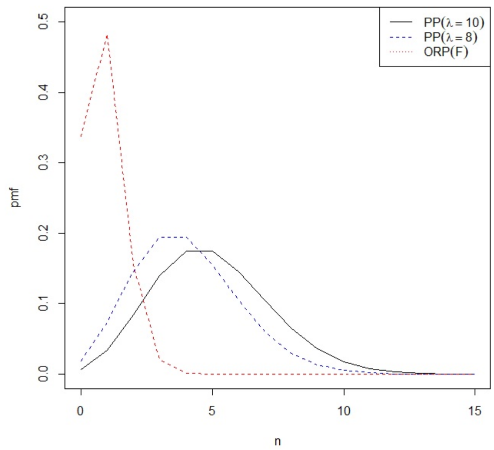

Fig. 1. pmf plots for $t = 0.5$

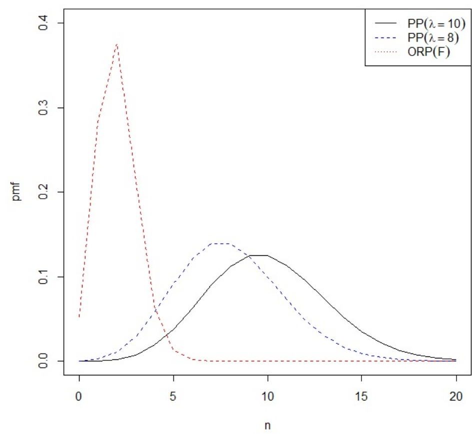

Fig. 2. pmf plots for $t = 1$

From (1.1) the probability mass function (pmf) for the rv $N_F(t)$ is obtained as

$$\mathrm{P}(N_F(t) = n) = \mathrm{P}(N_F(t) \geq n) - \mathrm{P}(N_F(t) \geq n + 1) = F_n(t) - F_{n+1}(t) \tag{1.2}$$

For a HPP $\{N(t), t \geq 0\}$ with rate λ, $N(t)$ is a Poisson(λt) rv and therefore the pmf for $N(t)$ is $\mathrm{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$.

The pmf plots of the Poisson rv $N(t)$ associated with rates $\lambda = 10$ and $\lambda = 8$, and for the counting rv $N_F(t)$ associated with ORP generated by Gamma distributed inter-arrival times with shape 2.2 and scale 5 are displayed in Figs. 1-3 for different values of time t. It is observed that as the time t increases the pmf plots shifted towards right which indicates that more arrivals are occurring which is desirable although the shape of the plots remains almost same. It is worth to note that while comparing the pmf plots for any value of t, the shape of the plots are varying based on the generating df of the inter-arrival times.

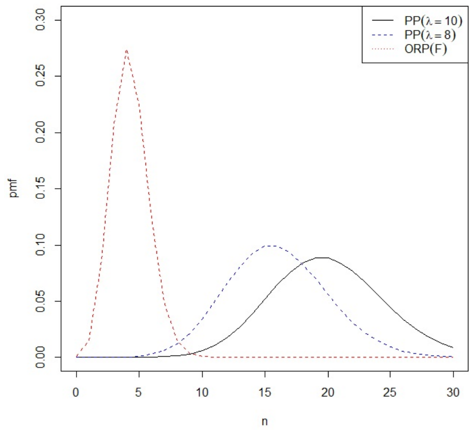

Fig. 3. pmf plots for $t = 2$

Renewal Function for an ORP

Renewal function (RNF) for the ORP $\{N_F(t), t \geq 0\}$, denoted as $M(t)$, is defined as the expected number of arrivals (failures) by time t. Therefore

$$M(t) = \mathrm{E}\left(N(t)\right) = \sum_{n=0}^{\infty} \mathrm{P}\left(N(t) > n\right) = \sum_{n=1}^{\infty} \mathrm{P}\left(N(t) \geq n\right) = \sum_{n=1}^{\infty} \mathrm{P}\left(S_n \leq t\right) = \sum_{n=1}^{\infty} F_n(t) \tag{1.3}$$

Remark 1.2. For an HPP (1.1), $M(t) = \lambda t$ as $N(t) \sim$ Poisson (λt).

The analytical form of RNF can be obtained by Laplace transform (LT) method as given below. Taking LT of both sides of (1.3), we get

$$\hat{M}(s) = \sum_{n=1}^{\infty} \hat{F_n}(s) = \frac{1}{s} \sum_{n=1}^{\infty} \hat{f_n}(s) = \frac{1}{s} \sum_{n=1}^{\infty} \left(\hat{f}(s)\right)^n = \frac{\hat{f}(s)}{s\left(1 - \hat{f}(s)\right)} \tag{1.4}$$

where, $f_n(t) = {F_n}'(t)$ be the n-fold convolution of f with itself, $f(t) = F'(t)$, and $\hat{g}(s)$ denote the LT of a function $g(t)$. Inverting (1.4), the closed form of $M(t)$ can be obtained for some specific baseline renewal dfs.

Renewal Intensity Function

Renewal density, also known as Renewal intensity function (RNIF), for the ORP $\{N_F(t), t \geq 0\}$, denoted as $m(t)$, is defined as

$$m(t) = M'(t) = \sum_{n=1}^{\infty} f_n(t) \tag{1.5}$$

Therefore $m(t)$ can be interpreted as the mean number of renewals to be expected in a narrow interval near t. Taking LT of both sides of (1.5), we get

$$\hat{m}(s) = s\hat{M}(s) = \frac{\hat{f}(s)}{1 - \hat{f}(s)} \tag{1.6}$$

and inverting (1.6) we get the analytical expression for $m(t)$.

Some numerical illustrations for the comparison of RNF and RNIF of independent ORPs are provided in Section (3), Example (3.5).

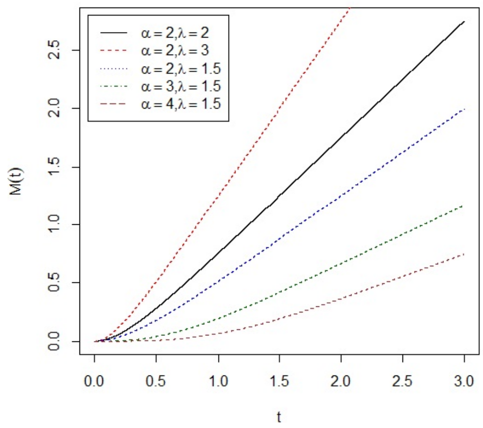

Fig. 4. Renewal function plots for Gamma renewal distribution

Remark 1.3. For an HPP (1.1), $m(t) = \lambda$, that is the RNIF for a HPP is constant.

Definition 1.3. A stochastic process $\{W(t), t \geq 0\}$ is said to be a compound renewal process (CRP) if it can be represented, for $t \geq 0$, by $W(t) = \sum_{i=1}^{N_F(t)} W_i$ where $\{N_F(t), t \geq 0\}$ is a ORP with renewal distribution F, and $\{W_i, i = 1, 2, \ldots\}$ is a family of iid rvs, each having df H, that is independent of the ORP $\{N_F(t), t \geq 0\}$. Therefore, the rv $W(t)$ can be interpreted as the total amount of time spent by time $t > 0$ where the arrivals (visits to the website) follow a ORP $\{N_F(t), t \geq 0\}$ and the each visitor spends a random amount of time at the site independent of the others, denoted as sequence of iid rvs $W_1, W_2, \ldots$, having df H.

For the CRP $\{W(t), t \geq 0\}$ as defined above, the df for the rv $W(t)$ is obtained as

$$\begin{aligned} \mathrm{P}(W(t) \leq x) &= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{N_F(t)} W_i \leq x | N_\lambda(t) = n\right) \mathrm{P}\left(N_F(t) = n\right) \\ &= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{n} W_i \leq x\right) \mathrm{P}\left(N_F(t) = n\right) \\ &= \sum_{n=1}^{\infty} H_n(x) \left(F_n(t) - F_{n+1}(t)\right), \text{ using (1.2)} \end{aligned} \tag{1.7}$$

where H_n denotes n-fold convolution of H with itself.

2 Main Results

Consider two independent ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$, generated respectively by two iid inter-arrival times $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ with respective dfs F and G. We state and prove our main results now. The following result characterizes the inter-arrival times of two independent ORPs by means of stochastic ordering and the comparison between two independent HPPs, a HPP and an independent ORP and characterization of inter-arrival times of two independent ORPs by means of hazard rate orders and likelihood ratio ordering, are obtained as corollaries to this result. We state and prove a Lemma (2.1) which will be used prove other results in this section.

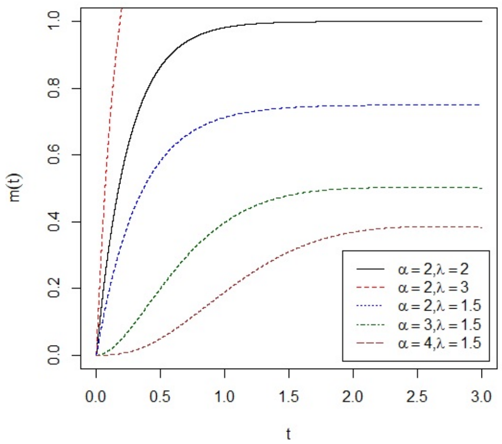

Fig. 5. Renewal intensity function plots for Gamma renewal distribution

Lemma 2.1. For two continuous non-negative rvs X and Y having dfs F and G respectively, $X \geq_{st} Y \Leftrightarrow F_n(t) \leq G_n(t)$ for $t \geq 0$, where F_n and G_n denotes n-fold convolution of F and of G with itself respectively.

Proof. $X \geq_{st} Y \Rightarrow \mathrm{P}(X > t) \geq \mathrm{P}(Y > t)$ for all $t \geq 0 \Rightarrow F(t) \leq G(t)$ for all $t \geq 0$. Therefore, $F_2(t) = F * F(t) = \int_0^t F(u)F(t-u)du \leq \int_0^t G(u)G(t-u)du = G*G(t) = G_2(t)$. Hence, the proposition is true for $n = 2$. Let the proposition true for $n-1$ for some positive integer $n > 2$, that is $F_{n-1}(t) \leq G_{n-1}(t), t \geq 0$. Therefore, $F_n(t) = F_{n-1} * F(t) = \int_0^t F_{n-1}(u)F(t-u)du \leq \int_0^t G_{n-1}(u)G(t-u)du = G_n(t)$. Hence, the proof follows using method of induction. The converse is trivial and therefore omitted. □

Theorem 2.2. The inter-arrival times of an ORP $\{N_F(t), t \geq 0\}$ is stochastically larger (smaller) than the inter-arrival times of another independent ORP $\{N_G(t), t \geq 0\}$ iff $N_F(t) \leq_{st} (\geq_{st}) N_G(t), t \geq 0$.

Proof. Since the inter-arrival times of ORP $\{N_F(t), t \geq 0\}$ are stochastically larger (smaller) than those of another independent ORP $\{N_G(t), t \geq 0\}$, for $t \geq 0$,

$$\begin{aligned} \bar{F}(t) \geq (\leq)\bar{G}(t) \quad &\Leftrightarrow \quad F(t) \leq (\geq) G(t) \\ &\Rightarrow \quad F_n(t) \leq (\geq) G_n(t), n \geq 0, \text{ using Lemma 2.1} \\ &\Rightarrow \quad \mathrm{P}\left(N_F(t) \geq n\right) \leq (\geq) \mathrm{P}\left(N_G(t) \geq n\right), n \geq 0, \text{ using (1.1)} \\ &\Rightarrow \quad N_F(t) \leq_{st} (\geq_{st}) N_G(t), \end{aligned}$$

proving necessity. To prove sufficiency,

$$N_F(t) \leq_{st} (\geq_{st}) N_G(t), t \geq 0 \Rightarrow \mathrm{P}\left(N_F(t) \geq n\right) \leq (\geq) \mathrm{P}\left(N_G(t) \geq n\right), n \geq 0, t \geq 0.$$

For $n = 1$, we get, for $t \geq 0$,

$$\begin{aligned} \mathrm{P}\left(N_F(t) \geq 1\right) \leq (\geq) \mathrm{P}\left(N_G(t) \geq 1\right) \quad &\Rightarrow \quad \mathrm{P}\left(N_F(t) < 1\right) \geq (\leq) \mathrm{P}\left(N_G(t) < 1\right) \\ &\Rightarrow \quad \bar{F}(t) \geq (\leq) \bar{G}(t). \end{aligned}$$

That is, the inter-arrival times of $N_F(.)$ are stochastically larger (smaller) than those of $N_G(.)$. Hence the proof. □

Corollary 2.3. Consider two independent HPPs $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ with respective rates $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$, where $\mu_1 > 0, \mu_2 > 0$. Then $N_1(t) \leq_{st} (\geq_{st}) N_2(t)$ iff $\mu_1 \geq (\leq) \mu_2$.

Proof. We have

$$\begin{aligned} \mu_1 \geq (\leq) \mu_2 \quad &\Leftrightarrow \quad \frac{t}{\mu_1} \leq (\geq) \frac{t}{\mu_2}, t \geq 0 \\ &\Leftrightarrow \quad e^{-\frac{t}{\mu_1}} \geq (\leq) e^{-\frac{t}{\mu_2}}, t \geq 0 \\ &\Leftrightarrow \quad N_1(t) \leq_{st} (\geq_{st}) N_2(t), t \geq 0, \end{aligned}$$

using Theorem 2.2. □

Corollary 2.4. Consider a HPP $\{N(t), t \geq 0\}$ with rate $\frac{1}{\mu}, \mu > 0$, and an independent ORP $\{N_F(t), t \geq 0\}$. Then $N_F(t) \leq_{st} (\geq_{st}) N(t)$ iff $R(t) \leq (\geq) \frac{t}{\mu}, t \geq 0$, where $R(.)$ is the hazard function associated with the df $F(\cdot)$.

Proof. Since $R(t) = -\log \bar{F}(t)$, for $t \geq 0$,

$$\begin{aligned} R(t) \leq (\geq) \frac{t}{\mu} \quad &\Leftrightarrow \quad -\log \bar{F}(t) \leq (\geq) \frac{t}{\mu} \\ &\Leftrightarrow \quad \bar{F}(t) \geq (\leq) e^{-\frac{t}{\mu}} \\ &\Leftrightarrow \quad N_F(t) \leq_{st} (\geq_{st}) N(t), \end{aligned}$$

using Theorem 2.2. □

Corollary 2.5. If the inter-arrival times of an ORP $\{N_F(t), t \geq 0\}$ is larger (smaller) than the inter-arrival times of another independent ORP $\{N_G(t), t \geq 0\}$ in the hazard rate order (reversed hazard rate order), then $N_F(t) \leq_{st} (\geq_{st}) N_G(t), t \geq 0$.

In other words, if $r_F(\cdot)$ and $r_G(\cdot)$ denote the respective hazard rate functions corresponding to the generating renewal dfs of the independent ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ such that $r_F(t) \leq (\geq) r_G(t)$ $(q_F(t) \geq (\leq)\ q_G(t))$, where $q_F(\cdot)$ and $q_G(\cdot)$ denote the respective reversed hazard rate functions), then $N_F(t) \leq_{st} (\geq_{st}) N_G(t), t \geq 0$.

Proof. Since the inter-arrival times of $\{N_F(t), t \geq 0\}$ being larger than the inter-arrival times of $\{N_G(t), t \geq 0\}$ in the hazard rate order (reversed hazard rate order) imply that the inter-arrival times of $\{N_F(t), t \geq 0\}$ are stochastically larger than those of $\{N_G(t), t \geq 0\}$, the proof follows from Theorem 2.2. □

Corollary 2.6. If the inter-arrival times of a renewal process $\{N_F(t), t \geq 0\}$ are larger (smaller) than those of another independent renewal process $\{N_G(t), t \geq 0\}$ in the likelihood ratio order, then $N_F(t) \leq_{st} (\geq_{st}) N_G(t), t \geq 0$.

Proof. Since the inter-arrival times of $\{N_F(t), t \geq 0\}$ being larger than those of $\{N_G(t), t \geq 0\}$ in the likelihood ratio order imply that the inter-arrival times of $\{N_F(t), t \geq 0\}$ are stochastically larger than those of $\{N_G(t), t \geq 0\}$, the proof follows from Theorem 2.2. □

The following result compares an ORP with that generated by the equilibrium distribution. We refer to the Appendix for the definitions of NBUE and NWUE.

Theorem 2.7. If $\left\{\tilde{N}_F(t), t \geq 0\right\}$ denotes the equilibrium F-renewal process, then $\tilde{N}_F(t) \geq_{st} N_F(t)$ iff F is NBUE, and the inequality reverses iff F is NWUE.

Proof. We prove the NBUE case and the NWUE case is similar. With $\mu_F = \int_0^\infty \bar{F}(x)dx < \infty$, by definition, the equillibrium df of F is $F_e(t) = \frac{1}{\mu_F}\int_0^t \bar{F}(x)dx, t \geq 0$, and $\bar{F}_e(t) = \frac{1}{\mu}\int_t^\infty \bar{F}(x)dx$. The delayed renewal process with the first inter-arrival time df as F_e and the later inter-arrival times being iid with df F is, by definition, the equilibrium F-renewal process $\left\{\tilde{N}_F(t), t \geq 0\right\}$. If F is NBUE, then, by definition, $\frac{1}{\mu_F}\int_t^\infty \bar{F}(x)dx \leq \bar{F}(t), t \geq 0$ so that $\bar{F}_e(t) \leq \bar{F}(t), t \geq 0$. Now $\mathrm{P}\left(N_F(t) \geq n\right) = F_n(t), t \geq 0, F_n$ the n-fold convolution of F with itself and $\mathrm{P}\left(\tilde{N}_F(t) \geq n\right) = F_e * F_{n-1}(t)$. We have

$$\begin{aligned} F_e(t) \geq F(t), t \geq 0 \quad &\Rightarrow \quad F_e * F_{n-1}(t) \geq F_n(t), t \geq 0, n \geq 2, \\ &\Rightarrow \quad \mathrm{P}\left(\tilde{N}_F(t) \geq n\right) \geq \mathrm{P}\left(N_F(t) \geq n\right), t \geq 0, n \geq 1, \text{ using (1.1)} \\ &\Rightarrow \quad \tilde{N}_F(t) \geq_{st} N_F(t), t \geq 0, \end{aligned}$$

proving sufficiency. For proving necessity, let $\tilde{N}_F(t) \geq_{st} N_F(t), t \geq 0$, so that $\mathrm{P}\left(\tilde{N}_F(t) \geq n\right) \geq \mathrm{P}\left(N_F(t) \geq n\right), n \geq 1, t \geq 0$. Since $\mathrm{P}\left(\tilde{N}_F(t) \geq 0\right) = \mathrm{P}\left(N_F(t) \geq 0\right) = 1, t \geq 0$, we have

$$\begin{aligned} \mathrm{P}\left(\tilde{N}_F(t) = 0\right) \leq \mathrm{P}\left(N_F(t) = 0\right), t \geq 0 \quad &\Rightarrow \quad \bar{F}_e(t) \leq \bar{F}(t), t \geq 0 \\ &\Rightarrow \quad \frac{1}{\mu_F}\int_t^\infty \bar{F}(x)dx \leq \bar{F}(t), t \geq 0 \\ &\Rightarrow \quad F \text{ is NBUE.} \end{aligned}$$

□

Remark 2.1. The following result combines Proposition 9.6.1 and Theorem 9.6.4 in Ross [1] and compares an ORP with a HPP.

If $\{N_F(t), t \geq 0\}$ is an ORP having mean inter-arrival time μ and $\{N(t), t \geq 0\}$ is a HPP with rate $\frac{1}{\mu}$, and if F is NBUE, then $F \leq_v Exp(\mu)$, the exponential df with mean μ, and $N_F(t) \leq_v N(t)$. The inequality reverses if F is NWUE.

The following result compares two ORPs under variability ordering.

Theorem 2.8. Consider two independent ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ having the same mean inter-arrival time $\mu, 0 < \mu < \infty$. If F is NBUE and G is NWUE, then $N_F(t) \leq_v N_G(t)$.

Proof. If $\{N(t), t \geq 0\}$ is a HPP with rate $\frac{1}{\mu}$, and F is NBUE, by Remark 2.1, we get $N_F(t) \leq_v N(t)$ and if G is NWUE, by Remark 2.1, we get $N_G(t) \geq_v N(t)$. Therefore, $N_F(t) \leq_v N(t) \leq_v N_G(t) \Rightarrow N_F(t) \leq_v N_G(t)$, completing the proof. □

Theorem 2.9. Let $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ be two independent ORPs having the same mean $\mu, 0 < \mu < \infty$, and $F \geq_v G$. Then $N_F(t) \leq_{st} N_G(t)$ iff F is NBUE and G is NWUE.

Proof. We have, for $t \geq 0$,

$$\begin{aligned} F \geq_v G \quad &\Rightarrow \quad \int_t^\infty \bar{F}(x)dx \geq \int_t^\infty \bar{G}(x)dx, \\ &\Rightarrow \quad \frac{1}{\mu}\int_t^\infty \bar{F}(x)dx \geq \frac{1}{\mu}\int_t^\infty \bar{G}(x)dx \\ &\Rightarrow \quad \bar{F}_e(t) \geq \bar{G}_e(t), \end{aligned}$$

and

$$F \text{ NBUE } \Rightarrow \frac{1}{\mu}\int_t^\infty \bar{F}(x)dx \leq \bar{F}(t) \Rightarrow \bar{F}_e(t) \leq \bar{F}(t),$$

$$G \text{ NWUE } \Rightarrow \frac{1}{\mu}\int_t^\infty \bar{G}(x)dx \geq \bar{G}(t) \Rightarrow \bar{G}_e(t) \geq \bar{G}(t).$$

Therefore

$$\bar{F}(t) \geq \bar{F}_e(t) \geq \bar{G}_e(t) \geq \bar{G}(t), t \geq 0 \Rightarrow F(t) \leq F_e(t) \leq G_e(t) \leq G(t), t \geq 0.$$

Also, $F(t) \leq G(t) \Rightarrow F_n(t) \leq G_n(t), n \geq 1$, by Lemma (2.1) so that $\mathrm{P}\left(\tilde{N}_F(t) \geq n\right) = F_e * F_{n-1}(t) \leq G_e * G_{n-1}(t) = \mathrm{P}\left(\tilde{N}_G(t) \geq n\right)$. Hence $\mathrm{P}\left(\tilde{N}_F(t) \geq n\right) \leq \mathrm{P}\left(\tilde{N}_G(t) \geq n\right), n \geq 0 \Rightarrow \tilde{N}_F(t) \leq_{st} \tilde{N}_G(t), t \geq 0$. Using Theorem 2.7, $\tilde{N}_F(t) \geq_{st} N_F(t)$ iff F is NBUE and $\tilde{N}_G(t) \leq_{st} N_G(t)$ iff G is NWUE. Combining these inequalities, we get

$$N_F(t) \leq_{st} \tilde{N}_F(t) \leq_{st} \tilde{N}_G(t) \leq_{st} N_G(t)$$

iff F is NBUE and G is NWUE, completing the proof. □

A generalized version of Theorem 2.9 is the following result.

Theorem 2.10. Let $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ be two independent ORPs having finite means μ_F and μ_G with $\mu_F \leq \mu_G$ and $F \geq_v G$. Then $N_F(t) \leq_{st} N_G(t)$ iff F is NBUE and G is NWUE.

Proof. The theorem can be proved using arguments similar to those used to prove Theorem 2.9 and hence the proof is omitted. □

Remark 2.2. Note that for two ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$, $F \leq_v G$ does not necessarily imply $N_F(t) \leq_v N_G(t)$. This can be seen from the following examples.

Consider two independent ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ having the same mean inter-arrival time $\mu, 0 < \mu < \infty$. If F is NBUE and G is NWUE, then by Theorem 2.8, $N_F(t) \leq_v N_G(t)$. Consider now an independent HPP $\{N(t), t \geq 0\}$ with rate $\frac{1}{\mu}$. Then F NBUE $\Rightarrow F \leq_v Exp(\mu)$ and G NWUE $\Rightarrow G \geq_v Exp(\mu)$ by Remark 2.1 which imply that $F \leq_v Exp(\mu) \leq_v G$. Therefore, with F NBUE and G NWUE, we have $F \leq_v G \Rightarrow N_F(t) \leq_v N_G(t)$. Consider two independent ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ with $F(t) = 1 - e^{-\frac{t}{\mu_F}}$, and $G(t) = 1 - e^{-\frac{t}{\mu_G}}$, $0 < \mu_F \leq \mu_G < \infty$. We then have

$$\begin{aligned}
\mu_F \leq \mu_G \quad &\Rightarrow \quad \frac{x}{\mu_F} \geq \frac{x}{\mu_G}, x \geq 0 \\
&\Rightarrow \quad e^{-\frac{x}{\mu_F}} \leq e^{-\frac{x}{\mu_G}}, x \geq 0 \\
&\Rightarrow \quad \int_t^\infty e^{-\frac{x}{\mu_F}}dx \leq \int_t^\infty e^{-\frac{x}{\mu_G}}dx, t \geq 0 \\
&\Rightarrow \quad \int_t^\infty \bar{F}(x)dx \leq \int_t^\infty \bar{G}(x)dx, t \geq 0 \\
&\Rightarrow \quad F \leq_v G,
\end{aligned}$$

but $\mathrm{E}(N_F(t)) = \frac{t}{\mu_F}$ and $\mathrm{E}(N_G(t)) = \frac{t}{\mu_G}$ so that $\mathrm{E}(N_F(t)) \geq \mathrm{E}(N_G(t)), t \geq 0$, which implies $N_F(t) \leq_v N_G(t)$ does not hold as $N_F(t) \leq_v N_G(t)$ implies that $\mathrm{E}(N_F(t)) \leq \mathrm{E}(N_G(t))$. The following results provide comparison of two independent CRPs by means of stochastic ordering on the basis of the df of iid summand rvs and/ or the renewal distribution of the corresponding ORP. Let us define the notations for use in Theorem 2.11 and Remark 2.3.

Consider two independent CPPs $\{A(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ defined as

$$A(t) = \sum_{i=1}^{N_F(t)} A_i, B(t) = \sum_{i=1}^{N_G(t)} B_i \tag{2.1}$$

where $\{A_i, i = 1, 2, \ldots\}$ and $\{B_i, i = 1, 2, \ldots\}$ are family of iid rvs, independent of the corresponding ORPs $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ respectively, and the iid rvs A_i and B_i are having dfs K and L respectively.

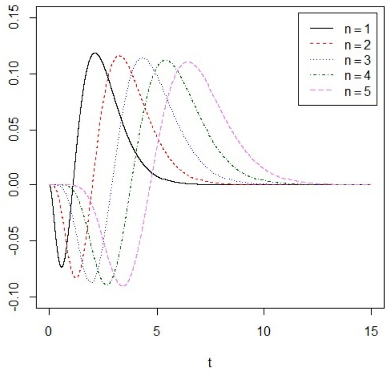

Fig. 6. $\mathrm{P}\left(N_F(t)=n\right)-\mathrm{P}\left(N_G(t)=n\right)$ **with** F **as Gamma(2,2) and** G **as Gamma(2,2.5)**

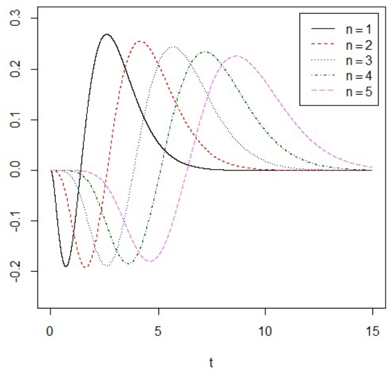

Fig. 7. $\mathrm{P}\left(N_F(t)=n\right)-\mathrm{P}\left(N_G(t)=n\right)$ **with** F **as Gamma(3,2) and** G **as Gamma(2,2)**

Theorem 2.11. *$A_i \geq_{st} B_i \Rightarrow A(t) \geq_{st} B(t), t \geq 0$ provided $F=G$.*

Proof. $A_i \geq_{st} B_i \Rightarrow K(x) \leq L(x), x \geq 0 \Rightarrow K_n(x) \leq K_n(x), x \geq 0$ using Lemma (2.1). Therefore using (1.7) and the condition $F=G$, we obtain

$$\mathrm{P}(A(t) \leq x)=\sum_{n=1}^{\infty} K_n(x)\left(F_n(t)-F_{n+1}(t)\right) \leq \sum_{n=1}^{\infty} L_n(x)\left(G_n(t)-G_{n+1}(t)\right)=\mathrm{P}(B(t) \leq x) \tag{2.2}$$

which implies $A(t) \geq_{st} B(t)$. □

Remark 2.3. It is to be noted that the condition $F=G$ is necessary for the result in Theorem (2.11) to be valid. This is because, if $F \neq G$, then $\mathrm{P}\left(N_F(t)=n\right) \leq \mathrm{P}\left(N_G(t)=n\right)$ for some values of n, and $\mathrm{P}\left(N_F(t)=n\right) \geq \mathrm{P}\left(N_G(t)=n\right)$ for some other values of n, that is the difference $\mathrm{P}\left(N_F(t)=n\right)-\mathrm{P}\left(N_G(t)=n\right)$ may be negative or positive for different values of n. This is illustrated in Figs. 6-8. where a plot of $\mathrm{P}\left(N_F(t)=n\right)-\mathrm{P}\left(N_G(t)=n\right)$ is shown for F and G as Gamma dfs with different shape and rate parameters and for $n=1,\ldots,5$. It is clear from the figure that the values of the difference is negative for some values of t and n and positive for other values of t and n. Therefore, if $F \neq G$, then the inequality in (2.2) may not hold.

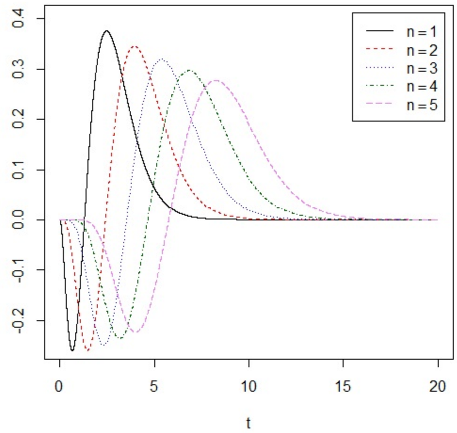

Fig. 8. $\mathrm{P}\left(N_F(t)=n\right)-\mathrm{P}\left(N_G(t)=n\right)$ **with** F **as Gamma(3,2) and** G **as Gamma(2,2.5)**

3 Illustrations

The results are illustrated here with examples keeping the same notations used in the previous section.

Example 3.1. Consider an ORP $\{N_F(t), t \geq 0\}$ having iid inter-arrival times with df F. and another independent ORP $\{N_G(t), t \geq 0\}$ with $G(x) = (F(x))^\theta$, where $\theta > 0$, is a parameter. If $0 < \theta < 1$, $F(x) \leq (F(x))^\theta = G(x), x \geq 0$ so that $\bar{F}(x) \geq \bar{G}(x), x \geq 0$, and by Theorem 2.2, $\{N_F(t), t \geq 0\} \leq_{st} \{N_G(t), t \geq 0\}$. If $\theta \geq 1$, $F(x) \geq (F(x))^\theta = G(x), x \geq 0$ so that $\bar{F}(x) \leq \bar{G}(x), x \geq 0$ and by Theorem 2.2, $\{N_F(t), t \geq 0\} \geq_{st} \{N_G(t), t \geq 0\}$.

Example 3.2. Consider two independent ORPs generated by dfs $F(t) = 1-e^{-t}, t \geq 0$, and $G(t) = 1-e^{-t^2}, t \geq 0$. If $0 \leq t < 1$, then $t \geq t^2$ and hence $\bar{F}(t) \leq \bar{G}(t)$ so that $N_F(t) \geq_{st} N_G(t)$ by Theorem 2.2. If $t \geq 1$, then $t \leq t^2$ and hence $\bar{F}(t) \geq \bar{G}(t)$ so that $N_F(t) \leq_{st} N_G(t)$ by Theorem 2.2.

Example 3.3. Consider an ORP generated by the df $F(t) = 1 - e^{-t^2}, t \geq 0$ with mean inter-arrival time $\mu_F = \int_0^\infty \bar{F}(t)dt = \frac{\sqrt{\pi}}{2}$, and an independent Poisson process $\{N(t), t \geq 0\}$ with rate $\frac{1}{\mu_F}$. Then the pdf of F is $f(t) = 2te^{-t^2}$ and $(\log f(t))'' = -\frac{1}{t^2} - 2 < 0, t \geq 0$ so that $f(.)$ is log-concave. Hence $F(.)$ is also log-concave (see Bagnoli and Bergstrom, 2005). Using the ageing class relationships, log-concave $\Rightarrow$ IFR $\Rightarrow$ NBU $\Rightarrow$ NBUE, F is NBUE, by Remark 2.1, $N_F(t) \leq_v N(t)$.

Example 3.4. Consider two independent CRPs $\{A(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ with $L(x) = (K(x))^\theta$ where $\theta > 0$ is a parameter. If $0 < \theta < 1$, $K(x) \leq (K(x))^\theta = L(x), x \geq 0$ and by Theorem 2.11, $A(t) \geq_{st} B(t)$. If $\theta \geq 1$, $K(x) \geq (K(x))^\theta = L(x), x \geq 0$ and by Theorem 2.11, $A(t) \leq_{st} B(t)$.

Example 3.5. Consider a ORP with Gamma renewal distribution, with α as shape parameter and λ as scale parameter. The LT of Gamma density $f(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}$ is obtained as $\hat{f}(s) = \left(\frac{\lambda}{\lambda+s}\right)^\alpha$. However the closed form of the RNF and RNIF is obtained only for positive integer shape parameter α. Using (1.4) and (1.6), for $\alpha = 2, 3,$ *and* 4 the RNFs and RNIFs are obtained as shown in Tables 1 and 2.

The RNF and RNIF plots for Gamma renewal distribution for different combinations of shape parameter $\alpha = 2, 3, 4,$ and scale parameter $\lambda = 1.5, 2, 3$ are displayed in Figs. 4 and 5. It can be observed from Fig. 4, that RNF $M(t)$ increases as t increases which is obvious, and for fixed $\alpha = 2$ and given $t > 0$, $M(t)$ increases as λ increases, whereas for fixed $\lambda = 1.5$ and given $t > 0$, $M(t)$ decreases as α increases.

Table 1. RNF

α	RNF
2	$\frac{\lambda t}{2} - \frac{1}{4} + \frac{e^{-2\lambda t}}{4}$
3	$\frac{\lambda}{3} + \frac{1}{3}\left(1 - e^{-\frac{3\lambda t}{2}}\left(\cos\left(\lambda t\sqrt{\frac{3}{4}}\right) + \frac{1}{\sqrt{3}}\sin\left(\lambda t\sqrt{\frac{3}{4}}\right)\right)\right)$
4	$\frac{\lambda t}{4} + \frac{e^{-2\lambda t}}{8} + \frac{e^{-\lambda t}(\cos(\lambda t)+\sin(\lambda t))}{4} - \frac{3}{8}$

Table 2. RNIF

α	RNIF
2	$\frac{\lambda}{2}\left(1 - e^{-2\lambda t}\right)$
3	$\frac{\lambda}{3}\left(1 - e^{-\frac{3\lambda t}{2}}\left(\cos\left(\lambda t\sqrt{\frac{3}{4}}\right) + \sqrt{3}\sin\left(\lambda t\sqrt{\frac{3}{4}}\right)\right)\right)$
4	$\frac{\lambda}{4}\left(1 - e^{-2\lambda t} - 2e^{-\lambda t}\sin(\lambda t)\right)$

From Fig. 5, we observe that the RNIF $m(t)$ is increasing rapidly near 0 as t increases, but gradually the plots become parallel to the time axis, that is RNIF tends to some constant as t getting larger. This is also evident from the analytical expression of the RNIFs. Also for fixed $\alpha = 2$ and given $t > 0$, $m(t)$ increases as λ increases, whereas for fixed $\lambda = 1.5$ and given $t > 0$, $m(t)$ decreases as α increases.

4 Conclusions

This paper has presented comparison of two independent ORPs by means of several stochastic orderings between the generating inter-arrival time rvs, like stochastic order, hazard rate order, likelihood ratio order and variability order, as well as on the basis of some ageing classes of the generating rvs, such as NBUE/ NWUE. Some stochastic ordering results obtained for the counting rvs associated with ORPs generated by some specific ageing life distributions. Some numerical illustrations are provided using gamma renewal distribution for comparing two independent ORPs using graph plots for the pmf of the counting rv, RNF and RNIF. We have also defined compound renewal process (CRP), as a generalization of compound Poisson process, and presented some stochastic comparisons of two independent CRPs on the basis of the iid summands and the renewal df of the generating ORPs.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Let X and Y be two nonnegative independent rvs with respective dfs $F(.)$ and $G(.)$, survival functions (sfs) $\bar{F}(.)$ and $\bar{G}(.)$, hazard rate functions $r_X(.)$ and $r_Y(.)$, reversed hazard rate functions $q_X(.)$ and $q_Y(.)$.

A few definitions:

1. For any continuous non-negative rv X, we define its hazard rate function and reversed hazard rate function as $r_X(t) = \frac{f(t)}{\bar{F}(t)}$ and $q_X(t) = \frac{f(t)}{F(t)}, t \geq 0$ respectively, where $f(.)$ denotes the probability density function (pdf) of the rv X.
2. X is said to be stochastically larger than Y, denoted by $X \geq_{st} Y$, if $\mathrm{P}(X > t) \geq \mathrm{P}(Y > t), t \geq 0$, or equivalently $F(t) \leq G(t), t \geq 0$.
3. X is said to be larger than Y in the hazard rate order, denoted by $X \geq_{hr} Y$, if $r_X(t) \leq r_Y(t), t \geq 0$, or equivalently if $\frac{\bar{F}(t)}{\bar{G}(t)}$ is increasing in t.
4. X is said to be larger than Y in the reverse hazard rate order, denoted by $X \geq_{rh} Y$ if $q_X(t) \geq q_Y(t), t \geq 0$, or equivalently if $\frac{F(t)}{G(t)}$ is increasing in t.
5. For two continuous non-negative independent rvs X and Y with respective pdfs f and g, X is larger than Y in the sense of likelihood ratio, denoted by $X \geq_{lr} Y$, if $\frac{f(t)}{g(t)} \uparrow t, t \geq 0$.
6. For two continuous non-negative independent rvs X and Y, X is said to be stochastically less variable than Y, denoted as $X \leq_v Y$, if $\int_t^\infty \bar{F}(x)dx \leq \int_t^\infty \bar{G}(x)dx, t \geq 0$.

   For two discrete non-negative independent rvs X and Y, X is said to be stochastically less variable than Y, denoted as $X \leq_v Y$, if $\sum_{x=n}^\infty \bar{F}(x) \leq \sum_{x=n}^\infty \bar{G}(x)$ for all $n = 0, 1, \ldots$.
7. X and the corresponding df F is said to be increasing failure rate (IFR) if $r_X(t) \uparrow t$.
8. X and the corresponding df F is said to be New Better than used (NBU) if $\bar{F}(s+t) \leq \bar{F}(t)\bar{F})(s), t \geq 0, s \geq 0$.
9. X and the corresponding df F is said to be New better than used in expectation (NBUE) if

   (a) X has finite mean $\mu_F = \int_0^\infty \bar{F}(x)dx$,
   (b) $\bar{F}(t) \geq \frac{1}{\mu_F} \int_t^\infty \bar{F}(x)dx, t \geq 0$.
10. A real valued function f is said to be concave (convex) if for any $x, y \geq 0$ and for any $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \geq (\leq)(1-\alpha)f(x) + \alpha f(y).$$

    If f is twice-differentiable, then f is concave (convex) iff $f^{''}$ is non-positive (non-negative).
11. f is said to be log-concave (log-convex) if $\log f$ is concave (convex).

The dual stochastic orders/ ageing classes are defined by reversing the inequalities.

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