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Bandwagon Distance and Bandwagon Eccentric Domination in Graphs

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this article, bandwagon distance is introduced and various parameters of bandwagon distance like bandwagon eccentricity, bandwagon eccentric vertex, bandwagon radius, bandwagon diameter, bandwagon center, bandwagon periphery are defined. Bounds on bandwagon radius and bandwagon diameter for class of graphs are found. Bandwagon eccentric domination is defined along with bandwagon eccentric domination number $\gamma_{bed}(G)$. Necessary and sufficient condition for bandwagon eccentric dominating set is proved. Results related to exact values of bandwagon eccentric domination number of class of graphs is obtained.

Keywords: Elected neighbour; bandwagon distance; bandwagon eccentricity; bandwagon eccentric domination.

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1 Introduction

The concept of distance in graph is very interesting. There are many different types of distances like geodesic, detour [1], steiner [2], D-distance [3], M-distance [4], eccentric, coupling [5], superior [6] etc. Some are metric and some are non-metric distance. 'Distance in graphs' by F. Buckley and F. Harary [7] is a textbook where some distances in graphs are discussed. Ore and Berge [8,9] introduced domination in graphs, there is a vast literature available in the textbooks [10-14]. T.N. Janakiraman et al. [15] introduced eccentric domination in graphs. There are many dominating sets whose constraints are based on distance parameters [16,17,18].

Bandwagon was the earliest modes of announcement used to communicate the orders, decisions, precautionary measures to people. A person used to announce the information in the streets. Later the term and wagon effect was adapted by psychologists to name a trend where a person's decisions were always influenced by the trend. People blindly followed the herd. People did this because they felt the idea was tried and tested. Therefore it was safe to follow and chances of failure was minimal.

Inspired by this concept bandwagon distance is introduced in this paper. We also introduce elected neighbour, bandwagon eccentricity, bandwagon eccentric vertex, bandwagon radius, bandwagon diameter, bandwagon center, bandwagon periphery, bandwagon eccentric domination. Results related to bandwagon eccentric domination of different family of graphs are stated and proved.

2 Preliminaries

Definition 2.1. [19] A walk W of a graph G is a finite alternating sequence of vertices and edges, begining and ending with vertices such that, each edge is incident with the vertices preceding and following it.

Definition 2.2. [19] The open neighborhood $N(u)$ of a vertex v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v.

Definition 2.3. [19] The vertex v is called isolated vertex if $deg(v) = 0$, the vertex u is called endpoint or pendant vertex if $deg(v) = 1$.

Definition 2.4. [20] The degree $deg(v)$ of v is the number of edges incident with v.

Definition 2.5. [15] The eccentricity $e(v)$ of v is the distance to a vertex farthest from v. Thus, $e(v) = max\{d(u, v) : u \in V\}$. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex. The Eccentric set of a vertex v is defined as $E(v) = \{u \in V(G) : d(u, v) = e(v)\}$.

Definition 2.6. [21] The radius of G is $rad(G) = min\{e(u) : u \in V(G)\}$ and the diameter of G is $diam(G) = max\{e(u) : u \in V(G)\}$.

Definition 2.7. [15] v is a central vertex if $e(v) = rad(G)$. The center $C(G)$ is the set of all central vertices. v is a peripheral vertex if $e(v) = diam(G)$. The periphery $P(G)$ is the set of all peripheral vertices. If $rad(G) = diam(G)$ then the graph is called self centered graph.

Definition 2.8. [15] A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric vertex of v in D.

Definition 2.9. [22] A friendship graph F_n can be constructed by joining $'n'$ C_3 cycle graph with a common vertex. The $'n'$ in F_n denotes the number of C_3 cycles in F_n. There are $2n + 1$ vertices and $3n$ edges in F_n graph. The graph F_1 is an exact copy of cycle C_3. The friendship graph F_2 is obtained by joining two C_3 cycles with a common vertex.

3. Bandwagon Distance in Graphs

In this section, we introduce bandwagon distance and discuss its parameters. Here only connected graphs are considered.

Definition 3.1. A vertex u is said to be an elected neighbour of v if u is adjacent to v and has the maximum degree among all vertices adjacent to v. The walk between any two vertices where all the vertices are connected to at least one of its elected neighbour is called bandwagon walk. The shortest bandwagon walk between any two vertices v_i and v_j is known as bandwagon distance given by $Bd(v_i, v_j)$.

Definition 3.2. The bandwagon eccentricity $Be(v)$ of a vertex v is the bandwagon distance to a vertex farthest from v. Thus, $Be(e) = max\{Bd(u,v) : v, u \in V\}$. For a vertex v, each vertex at a distance $Be(v)$ from v is a bandwagon eccentric vertex. The bandwagon eccentric set of a vertex v is defined by $BE(v) = \{u \in V(G) : d(u,v) = Be(e)\}$.

Definition 3.3. The bandwagon radius $Brad(G)$ is the minimum bandwagon eccentricity of the vertices. The bandwagon diameter $Bdiam(G)$ is the maximum bandwagon eccentricity. v is a bandwagon central vertex if $Be(v) = Brad(G)$. The bandwagon center $BC(G)$ is the set of all bandwagon central vertices. v is a bandwagon peripheral vertex if $Be(v) = Bdiam(G)$. The bandwagon periphery $BP(G)$ is the set of all bandwagon peripheral vertices. A graph G is said to be bandwagon self-centered if and only if $Brad(G) = Bdiam(G)$.

Example 3.1. Consider the graph G given in Fig. 1.

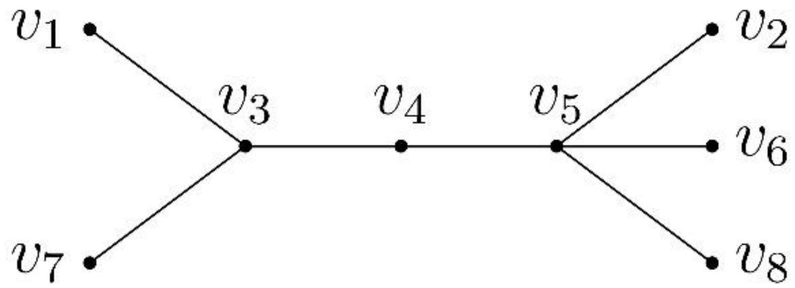


Fig. 1. Graph G

Table 1. Bandwagon eccentric vertex $BE(v)$

Vertex $v \in V(G)$	Bandwagon eccentricity $Be(v)$	Bandwagon eccentric vertex $BE(v)$
v_1	6	$\{v_7\}$
v_2	4	$\{v_1, v_6, v_7, v_8\}$
v_3	5	$\{v_1, v_7\}$
v_4	4	$\{v_1, v_7\}$
v_5	3	$\{v_1, v_2, v_6, v_7, v_8\}$
v_6	4	$\{v_1, v_2, v_7, v_8\}$
v_7	6	$\{v_1\}$
v_8	4	$\{v_1, v_2, v_6, v_7\}$

$Brad(G) = 3, Bdiam(G) = 6, BC(G) = \{v_5\}$ and $BP(G) = \{v_1, v_7\}$.

The bandwagon distance between v_1 and v_4, $(Bd(v_1, v_4))$ is 4. A bandwagon distance path is $v_1 - v_3 - v_4 - v_5 - v_4$, since the elected neighbour of v_4 is v_5. By the Definition-3.1, every vertex has to be connected to its elected neighbour.

Similarly the bandwagon distance between v_1 and v_3, ($Bd(v_1, v_3)$) is 5. The bandwagon distance path is $v_1 - v_3 - v_4 - v_5 - v_4 - v_3$, since the elected neighbour of v_3 is v_4 and the elected neighbour of v_4 is v_5.

Remark 3.1. For a graph K_1, $Bd(K_1) = 0$.

Theorem 3.1. For any connected graph G, where order of G is greater than or equal to 2. The following are true.
1. $Bd(v, v) \neq 0$.
2. $Bd(u, v) = Bd(v, u)$.
3. $Bd(u, v) \leq Bd(u, w) + Bd(w, v)$.

Proof:

1. Every bandwagon vertex has an elected neighbour on a bandwagon walk. Therefore the minimum possible bandwagon distance is 2. Therefore $Bd(v, v) > 0$.
2. Since every vertex on the bandwagon walk has an elected neighbour and it is the shortest bandwagon distance between (u, v) and vice versa, the bandwagon distance remains symmetric. Therefore $Bd(u, v) = Bd(v, u)$.
3. Let (u, v) be a bandwagon distance then $Bd(u, v) = Bd(v, u)$ it is symmetric.

Case(i): There exist a vertex w on the same walk between $Bd(u, v)$. Then $Bd(u, v) \leq Bd(u, w) + Bd(w, v)$.
Case(ii): w does not lie on the shortest $Bd(u, v)$ walk. Since $Bd(u, v)$ is shortest distance. The proof is obvious and $Bd(u, v) < Bd(u, w) + Bd(w, v)$.

Remark 3.2.

1. From the Theorem-3.1, bandwagon distance is not a metric.
2. Consider path graph with vertices $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where v_1 and v_n are pendant vertices, hence $deg(v_1) = deg(v_n) = 1$ and $deg(v_2) = deg(v_3) = \cdots = deg(v_{n-1}) = 2$. Therefore $Bd(u, v) = d(u, v) = 1$, if $u, v \in E(u, v) \in P_n$ and u, v not a pendant vertices.
3. For any star graph S_n $\forall\, n \geq 3$, where $u, v \in V(S_n)$

$$Bd(u, v) = \begin{cases} 1, if\ u, v \in E(u, v) \in S_n, \\ 2, if\ u, v \notin E(u, v) \in S_n. \end{cases}$$

4. If G is a complete, cycle, crown and cocktail party graphs, then $Bd(u, v) = d(u, v)\ \forall\, u, v \in G$ since they are regular graphs.

Observation 3.1. For any star graph S_n $\forall\, n \geq 3$, v_1 is the central vertex and all other vertices are pendant vertices then

1. $BE(v_1) = V(S_n) - \{v_1\}$ and $BE(v_i) = V(S_n) - \{v_1, v_i\}$ where $2 \leq i \leq n$.
2. $|BE(v_1)| = n - 1$ and $|BE(v_i)| = n - 2$.

Theorem 3.2. For any star graph S_n $\forall\, n \geq 3$, then

i. $Brad(S_n) = 1$ and $|BC(S_n)| = 1$.
ii. $Bdiam(S_n) = 2$ and $|BP(S_n)| = n - 1$.

Proof: (i): Every star graph S_n $\forall\, n \geq 3$ contains a central vertex v_1 and all other $(n - 1)$ vertices are pendant vertices. Then $deg(v_2) = deg(v_3) = \cdots = deg(v_n) = 1$. Now the bandwagon distance between the central vertex v_1 and pendant vertex v_i $\forall\, 2 \leq i \leq n$ is given by $Bd(v_1, v_i) = 1$, since all the pendant vertices has same degree. The bandwagon distance between any two pairs of pendant vertices is given by $Bd(v_i, v_j) = 2\ \forall\, i, j$, and $2 \leq i < j \leq n$, since every pendant vertex is adjacent to the central vertex v_1 which has the maximum degree. Therefore the central vertex v_1 becomes a vertex with a unique bandwagon eccentric value. Hence $Brad(S_n) = 1$ and $|BC(S_n)| = 1$.

(ii): By (i) $|BC(S_n)| = 1$, since v_1 being the unique vertex with the least bandwagon eccentric value and all the other remaining pendant vertices $v_i \in V(S_n) - \{v_1\} \forall i$ and $2 \leq i \leq n$ have the same bandwagon eccentricity $Be(v_i) = 2$. Therefore $Bdiam(S_n) = 2$ and $|BP(S_n)| = n - 1$. Hence, the set of all pendant vertices of S_n form the bandwagon periphery of S_n.

Observation 3.2. For any path graph $P_n \forall n \geq 3$,

1. The bandwagon center $BC(P_n)$ has unique vertex ie, $|BC(P_n)| = 1$ if $'n'$ is odd.
2. $BC(P_n)$ contains a pair of bandwagon central vertices ie, $|BC(P_n)| = 2$ if $'n'$ is even.
3. The bandwagon periphery $BP(P_n)$ has pendant vertices ie, $|BP(P_n)| = 2$.

Theorem 3.3. For any path graph P_n,

$$Brad(P_n) = \begin{cases} \dfrac{n-1}{2}, if\ 'n'\ is\ odd, n \geq 3, \\ \dfrac{n}{2}, if\ 'n'\ is\ even, n \geq 6. \end{cases}$$

Proof: Case(i): If n is odd, from the Observation-3.2-(1), any odd path P_n contains a unique vertex that forms the bandwagon center. Then the pendant vertices v_1 and v_n form the bandwagon eccentric vertices of the unique vertex v_i, which is also the bandwagon center. The bandwagon distance $Bd(v1, vi) = Bd(vi, vn) = \frac{n-1}{2}$. Hence $Brad(P_n) = \frac{n-1}{2}$, if n is odd and $n \geq 3$.

Case(ii): If n is even, from the Observation 3.2 (2), there is a pair of intermediate adjacent vertices which forms the bandwagon center of P_n. Let v_i and v_j be the intermediate adjacent vertices of the path $Brad(P_n) = \{v_i, v_{i+1}\}$, $i = \left\lfloor \frac{n}{2} \right\rfloor$ and $Be(v_i) = Be(v_{i+1})$. The bandwagon eccentric vertices of v_i and v_j, where $i < j$ are v_n and v_1 respectively. The bandwagon distance between the central vertices and the pendant vertices will be same. Hence $Bd(v_i, v_n) = Bd(v_j, v_n) = \frac{n}{2}$. Therefore $Brad(P_n) = \frac{n}{2}$, if n is even, $n \geq 6$.

Theorem 3.4. For any path graph $P_n \forall n \geq 3$, the bandwagon diameter is $Bdiam(P_n) = n - 1$.

Proof: Consider the path graph P_n where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. The maximum bandwagon distance between any two vertices can be found between the pendant vertices ie, $BD(v_1, v_n) = n - 1$. Since there is no other alternative path between the pendant vertices, hence $Be(v_1) = Be(v_n) = n - 1, BE(v_1) = \{v_n\}$ and $BE(v_n) = \{v_1\}$. Therefore $Bdiam(P_n) = n - 1, BP(P_n) = \{v_1, v_n\}$ and $|BP(P_n)| = 2$.

Observation 3.3. For any cycle graph $C_n \forall n \geq 3$,

1. $|BE(v_i)| = 1$, if n is even and $\forall\ 1 \leq i \leq n$.
2. $|BE(v_i)| = 2$, if n is odd and $\forall\ 1 \leq i \leq n$.
3. $Brad(C_n) = Bdiam(C_n)$ ie, it is a bandwagon self-centered graph.

Theorem 3.5. For any cycle graph $C_n \forall n \geq 3$,

$$Brad(C_n) = \begin{cases} \dfrac{(n-1)}{2}, if\ n\ is\ odd, n \geq 3, \\ \dfrac{n}{2}, if\ n\ is\ even, n \geq 4. \end{cases}$$

Proof: Case(i): Let C_n be a cycle graph where $'n'$ is odd. Even cycle graph is a 2-regular graph, then the bandwagon distance between any two vertices is same as its geodesic distance. For every vertex $v_i \in V(C_n) \forall i$ and $1 \leq i \leq n$ the farthest vertex lies at a distance of $\frac{n-1}{2}$ from it. Hence $e(v_i) = \frac{n-1}{2}$. Therefore the bandwagon radius is given by $Brad(C_n) = \frac{n-1}{2}$.

Case(ii): For an even cycle C_n, where $v_i \in V(C_n)\ \forall\ i, 1 \le i \le n$ the vertex farthest from v_i lies at a distance of $\frac{n}{2}$ from it ie, $Be(v_i) = \frac{n}{2}$. Therefore the bandwagon radius of C_n is given by $Brad(C_n) = \frac{n}{2}$. Hence for an even cycle which is also self-centered, $Brad(C_n) = Bdiam(C_n) = \frac{n}{2}$.

Observation 3.4.

1. The wheel graph W_4 is bandwagon self-centered graph, ie, $Brad(W_4) = Bdiam(W_4) = 1$. Since W_4 is a regular graph $Bd(u,v) = 1, \forall\ u,v \in V(W_4)$.
2. For any wheel graph $W_n\ \forall\ n \ge 5$, consider the vertex set $V(W_n) = \{v_1, v_2, \dots, v_n\}$, where v_1 is the central vertex then

$$Bd(u,v) = \begin{cases} 1, either\ u\ or\ v\ is\ a\ central\ vertex, \\ \qquad 2,\ otherwise. \end{cases}$$

3. For any wheel graph $W_n\ \forall\ n \ge 5$, consider the vertex set $V(W_n) = \{v_1, v_2, \dots, v_n\}$, where v_1 is the central vertex then

   $BE(v_1) = V(W_n) - \{v_1\}$ and $BE(v_i) = V(W_n) - \{v_1, v_i\}$ where $2 \le v_i \le n$.
   $|BE(v_1)| = n - 1$ and $|BE(u)| = n - 2$.

Theorem 3.6. For any wheel graph $W_n\ \forall\ n \ge 5$, then

i. $Brad(W_n) = 1$ and $|BC(W_n)| = 1$.
ii. $Bdiam(W_n) = 2$ and $|BP(W_n)| = n - 1$.

Proof: Let W_n be the wheel graph where $V(W_n) = \{v_1, v_2, \dots, v_n\}$.

(i) Let v_1 be the vertex adjacent to all other vertices of a graph. Therefore $deg(v_1) = n - 1$ and the degree of all other vertices of a wheel graph has a degree 3 ie, $deg(v_i) = 3$ where $v_i \in V(W_n) - \{v_1\}\ \forall\ 2 \le i \le n$. Since every vertex of $V(W_n) - \{v_1\}$ is incident on the vertex v_1, the bandwagon distance is $Bd(v_1, v_i) = 1, Be(v_1) = 1, BE(v_1) = V(W_n) - \{v_1\}$ and $|BE(v_1)| = n - 1$. Therefore $Brad(W_n) = 1$ and $|BC(W_n)| = 1$.

(ii) Now consider any two vertices v_i and v_j other than v_1 ie, $2 \le i < j \le n$ then the shortest bandwagon distance between them will be of path v_i – v_1 – v_j , since v_1 has the maximum degree among all the vertices, hence $Bd(v_i, v_j) = 2$. Therefore $Be(v_i) = 2\ \forall\ i \ne 1, BE(v_i) = V(W_n) - \{v_1, v_i\}$ and $|BE(v_i)| = n - 2$. Hence $Bdiam(W_n) = 2$.

Observation 3.5. For friendship graphs F_n where $n \ge 5$,

1. $|Brad(G)| = 1$.
2. $|Bdiam(G)| = n - 1$.
3. Let $deg(v_i) = \Delta(G)$, then $BE(u) = V(G) - \{v_i, u\}$ where $u \in V(G)$. $|BE(v_i)| = n - 1$ and $|BE(u)| = n - 5$.

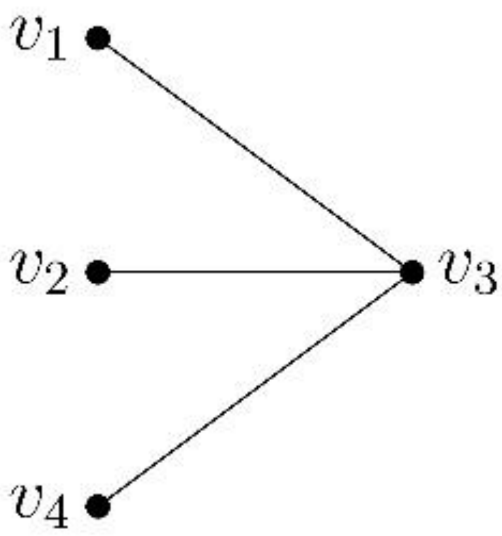
Theorem 3.7. For any friendship graphs F_n, then $Brad(G) = 1$ and $Bdiam(G) = 2$.

Proof: Let $v_1 \in V(F_n)$ be the common vertex of F_n which joins every cycle C_3 by a common point, then $deg(v_1) = 2n$. Since v_1 is adjacent to every other vertex $v_i \in V(F_n) - \{v_1\}, \forall\ 2 \le i \le (2n + 1), Bd(v_1, v_i) = 1, Be(v_1) = 1, BE(v_1) = V(F_n) - \{v_1\}$ and $|BE(v_1)| = n - 1$. Therefore $Brad(F_n) = 1$. Let $v_i, v_j \in V(F_n) - \{v_1\}$, since v_1 has the maximum degree and is adjacent to all other vertices, the bandwagon path between any two vertices should pass through the vertex v_1 ie, v_i – v_1 – v_j , $Bd(v_i, v_j) = 2$ since $deg(v_i) = 2\ \forall\ i \ne 1. Be(v_i) = 2, BE(v_i) = V(F_n) - \{v_1, v_i\}$ and $|BE(v_i)| = n - 2$. Therefore $Bdiam(F_n) = 2$.

4 Bandwagon Eccentric Domination

Definition 4.1. A dominating set $D \subseteq V(G)$ is a bandwagon eccentric dominating set (BED-set) if for every vertex $v \in V - D$, there exists at least one bandwagon eccentric vertex of v in D. A BED-set D is called a minimal BED-set if no proper subset of D is a BED-set. The BED-number $\gamma_{bed}(G)$ of a graph G is the minimum cardinality among the minimal BED-sets of G. The upper BED-number $\Gamma_{bed}(G)$ of a graph G is the maximum cardinality among the minimal BED-sets of G.

Example 4.1. Consider the claw graph given in Fig. 2.



Fi. 2. Claw graph

Here the dominating set is $\{v_3\}$, therefore $\gamma(G) = 1$ but it is not a BED-set. The BED-sets are $\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}$, therefore $\gamma_{bed}(G) = 2$. The upper BED-set is $\{v_1, v_2, v_4\}$, therefore $\Gamma_{bed}(G) = 3$.

Theorem 4.1. A bandwagon eccentric dominating set D is a minimal bandwagon eccentric dominating set if and only if for each vertex $u \in D$, one of the following is true.

1. u is an isolated vertex of D or u has no bandwagon eccentric vertex in D.
2. There exists some $u \in V - D$ such that $N(u) \cap D = u$.

Proof: Assume that D is a minimal bandwagon eccentric dominating set of G. Then for every vertex $u \in D$, $D - \{u\}$ is not a bandwagon eccentric dominating set. That is there exists some vertex v in $(V - D) \cup \{u\}$ which is not dominated by any vertex in $D - \{u\}$ or there exists v in $(V - D) \cup \{u\}$ such that v has no bandwagon eccentric vertex in $D - \{u\}$.

Case (i): Suppose $u = v$, then u is an isolate of D or u has no bandwagon eccentric vertex in D.
Case (ii): Suppose $v \in V - D$,

(a) If v is not dominated by $D - \{u\}$, but is dominated by some vertex in D, then v is adjacent to only u in D, that is $N(v) \cap D = \{u\}$.
(b) Suppose v has no bandwagon eccentric vertex in $D - \{u\}$ but v has a bandwagon eccentric vertex in D. Then u is the only bandwagon eccentric vertex of v in D. That is $E(v) \cap D = \{u\}$.

Conversely, suppose that D is a bandwagon eccentric dominating set and for each $u \in D$ one of the conditions holds, we show that D is a minimal bandwagon eccentric dominating set.

Suppose that D is not a minimal bandwagon eccentric dominating set, that is, there exists a vertex $u \in D$ such that $D-\{u\}$ is a bandwagon eccentric dominating set. Hence, u is adjacent to at least one vertex v in $D-\{u\}$ and u has a bandwagon eccentric point in $D-\{u\}$. Therefore, condition-(i) does not hold.

Also, if $D-\{u\}$ is a bandwagon eccentric dominating set, every element x in $V-D$ is adjacent to at least one vertex in $D-\{u\}$ and x has a bandwagon eccentric vertex in $D-\{u\}$. Hence, condition-(ii) does not hold.
This is a contradiction to our assumption that for each $u \in D$, one of the conditions holds. This proves the theorem.

Theorem 4.2. For star graph S_n, where $n \geq 3, \gamma_{bed}(S_n) = 2$.

Proof: Consider $V(S_n) = \{v_1, v_2, \ldots, v_n\}$, let v_1 be the central vertex and all the other vertices are pendant vertices, $deg(v_1) = n-1$ and $deg(v_i) = 1, \forall\, i\,, 2 \leq i \leq n$. Consider a set $D = \{v_1\}$ is the only dominating set of cardinality one, but it does not satisfy bandwagon eccentricity, $Be(v_1) = 1, BE(v_1) = V(S_n) - \{v_1\}, Be(v_i) = 2, \forall\, i\,, 2 \leq i \leq n, BE(v_i) = V(S_n) - \{v_1, v_i\}$ and $|BE(v_i)| = n-2$. Since every pendant vertex is a bandwagon eccentric vertex of each other. A set $D' = \{v_1, v_i\}, i \neq 1$ forms a bandwagon eccentric dominating set, since for every $v_j \in V(S_n) - D'$ there exists a vertex $v_i \in D'$ such that $BE(v_j) = v_i$ and v_1 dominates all the vertices. Therefore the set D' becomes the BED set and hence $\gamma_{bed}(S_n) = 2, \forall\, n \geq 3$.

Theorem 4.3. For wheel graph W_n, where $n \geq 5, \gamma_{bed}(W_n) = 2$.

Proof: Consider $V(W_n) = \{v_1, v_2, \ldots, v_n\}$, let v_1 be the central vertex ie, $deg(v_1) = n-1$ and $deg(v_i) = 3$ where $v_i \neq v_1$ is any other vertex of wheel graph. Suppose a set $D = \{v_1\}$ be the only dominating set of cardinality one, but it is not a bandwagon eccentric dominating set, since it does not satisfy the bandwagon eccentricity condition ie, $Be(v_1) = 1, BE(v_1) = V(W_n) - \{v_1\}, Be(v_i) = 2, BE(v_i) = V(W_n) - \{v_1, v_i\}$ and $|BE(v_i)| = n-2$. Since every vertex of degree three is a bandwagon eccentric vertex of each other. A set $D' = \{v_1, v_i\}, i \neq 1$ forms a bandwagon eccentric dominating set, since for every vertex $v_j \in V(W_n) - D'$ there exists a vertex $v_i \in D'$ such that $BE(v_j) = v_i$ and v_1 dominates all the vertices in the graph. Therefore the set D' becomes the BED set, hence $\gamma_{bed}(W_n) = 2, \forall\, n \geq 5$.
Note: $\gamma bed(W_4) = 1$.

Theorem 4.4. For Friendship graph F_n, where $n \geq 5$ and n is odd, $\gamma_{bed}(F_n) = 2$.

Proof: Consider $V(F_n) = \{v_1, v_2, \ldots, v_{2n+1}\}$ be the vertex set of a friendship graph containing $'n'$ number of C_3 cycle graphs. Let v_1 be the central vertex or common vertex of every cycle C_3 ie, $deg(v_1) = 2n$ and $deg(v_i) = 2, i = 2, 3, \ldots, (2n+1)$. $Be(v_1) = 1, BE(v_1) = V(F_n) - \{v_1\}, |BE(v_1)| = (2n+1) - 1 = 2n, Be(v_i) = 2, BE(v_i) = V(F_n) - \{v_1, v_i\}$ and $|BE(v_i)| = (2n+1) - 2 = (2n-1)$. Consider a set $D = v_1$ is the only dominating set with cardinality one, but it is not a bandwagon eccentric dominating set, since every vertex of degree two is a bandwagon eccentric vertex of each other. A set $D' = \{v_1, v_i\}, i = 2, 3, \ldots, (2n+1), i \neq 1$, forms a bandwagon eccentric dominating set, since for every vertex $v_j \in V(F_n) - D'$ there exists a vertex $v_i \in D'$ such that $BE(v_j) = v_i$ and v_1 dominates all the vertices in the graph. Therefore the set D' becomes the BED set and hence $\gamma_{bed}(F_n) = 2, \forall\, n = 5, 7, 9,$

Theorem 4.5. For cycle graph C_n, where $n \geq 4$ and n is even, $\gamma_{bed}(C_n) = n/2$.

Proof: Every cycle graphs of ever number of vertices is a self-centered graph ie, $Brad(C_n) = Bdiam(C_n)$, the bandwagon eccentricity of every vertex is same. Then for any vertex $v_i \in V(C_n)$, the diagonally opposite vertex will be the bandwagon eccentric vertex of v_i. Let D be the minimum dominating set containing $|D| = \left\lceil \frac{n}{3} \right\rceil$ vertices, but it is not a bandwagon eccentric dominating set. Therefore consider D with $\left(\frac{n}{3}\right)$ vertices, here D must not contain two vertices that are diagonally opposite to each other, since the diagonally opposite vertices are the bandwagon eccentric vertices to each other. Consider a set D such that the bandwagon distance between the vertices $v_i, v_j \in D$ is either 1, 2 or 3 ie, $Bd(v_i, v_j) = 1, 2$ or 3, these vertices forms a bandwagon eccentric dominating set. Since for every vertex $v_k \in V(C_n) - D$ there exists $BE(v_k) = v_i \in D$. Therefore $\gamma_{bed}(C_n) = \left(\frac{n}{2}\right), \forall\, n \geq 4, n$ is even.

Theorem 4.6. For cycle graph C_n, where $n \geq 3$ and n is odd, $k \in \mathbb{Z}^+$

$$\gamma_{bed}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, if\ n = 3k\ or\ n = 3k+1, \\ \left\lceil \frac{n}{3} \right\rceil + 1, \quad if\ n = 3k+2. \end{cases}$$

Proof: Case(i): For any cycle graphs with odd number of vertices and for every vertex $v_k \in V(C_n)$ has two bandwagon eccentric vertices v_i, v_j such that $BE(v_k) = \{v_i, v_j\}$. The eccentric vertices v_i, v_j will always be adjacent to v_k ie, $(v_i, v_j) \in E(C_n)$. The vertices v_i and v_j are placed at a distance of $\left(\frac{n-1}{2}\right)$ from v_k. Since every vertex dominates its adjacent vertices, $\left\lceil \frac{n}{3} \right\rceil$ set of vertices form, a dominating set of the cycle graph. The dominating set D forms the bandwagon eccentric dominating set, since for every vertex $v_k \in V(C_n) - D$ there exists at least one vertex $v_i \in D$ such that $BE(v_k) = v_i$. Therefore $\gamma_{bed}(C_n) = \left\lceil \frac{n}{3} \right\rceil$, if $n = 3k$ or $n = 3k + 1$.
Case(ii): For cycle graphs with odd number of vertices, if $n = 3k + 2$ then the cycle graphs are of the form $C_5, C_{11}, C_{17}, \ldots$ Similar to case(i), every vertex $v_k \in V(C_n)$ has two bandwagon eccentric vertices v_i and v_j, $BE(v_k) = \{v_i, v_j\}$ such that v_i and v_j are adjacent to v_k, $(v_i, v_j) \in E(C_n)$. Bandwagon eccentric vertices v_i and v_j are placed at a distance of $\left(\frac{n-1}{2}\right)$ from v_k. Since every vertex dominates its adjacent vertices, when $n = 3k + 2$ then $n = 5, 11, 17, \ldots, 3k + 2$. Therefore the cardinality of the bandwagon eccentric dominating set of a cycle graphs of the form C_{3k+2} is $\left\lceil \frac{n}{3} \right\rceil + 1$. Hence $\gamma_{bed}(C_n) = \left\lceil \frac{n}{3} \right\rceil + 1$, if $n = 3k + 2$.

Theorem 4.7. For path graph P_n, where $n \geq 2$,

$$\gamma_{bed}(P_n) = \begin{cases} \left(\frac{n+1}{3}\right) + 1, if\ n = 3k+2, k = 2,3, \ldots \\ \left\lceil \frac{n+1}{3} \right\rceil, \quad otherwise \end{cases}$$

Proof: Case(i): For $n = 3k + 2, \forall\ k = 2, 3, \ldots$, the path graphs are of the form $P_8, P_{11}, P_{14}, \ldots, P_{3k+2}$. Let D be a minimum dominating set of $P_{3k+2}, \forall\ k = 2, 3, \ldots$ then $\gamma(P_{3k+2}) = \left(\frac{n+1}{3}\right)$, but D is not a bandwagon eccentric dominating set. Since the dominating set D does not contain both the pendant vertices. To satisfy the condition of having a bandwagon eccentric vertex, the dominating set should contain both the pendant vertices of the path graph. Therefore the set D contains $\left(\frac{n+1}{3}\right) + 1$ vertices, hence for every vertex $v_j \in V(P_n) - D$ there exists at least one vertex $v_i \in D$ such that $BE(v_j) = v_i$. Therefore $\gamma_{bed}(P_{3k+2}) = \left(\frac{n+1}{3}\right) + 1, \forall\ k = 2, 3, \ldots$

Case(ii): The path graphs are of the form $P_2, P_3, P_4, P_5, P_6, P_7, P_9, P_{10}, P_{12}, \ldots$ The pendant vertices of the path form the bandwagon eccentric vertices ie, if $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, here v_1 and v_n are the pendant vertices then for any vertex $v_i \in V(P_n)$, v_1 or $v_n \in BE(v_i)$. Since the minimum number of vertices required to dominate a path graph is $\left(\frac{n+1}{3}\right)$ vertices, for certain path graphs this case gives fractional value, therefore the cardinality of the set D is given by $\left\lceil \frac{n+1}{3} \right\rceil$ and for every vertex $v_j \in V(P_n) - D$ there exists at least one vertex $v_i \in D$ such that $BE(v_j) = v_i$ which satisfies the condition of having bandwagon eccentricity vertex. Hence $\gamma_{bed}(P_n) = \left\lceil \frac{n+1}{3} \right\rceil$.

5 Conclusion

In this article the concept of bandwagon distance is introduced and various parameters of bandwagon distance like bandwagon eccentricity, bandwagon eccentric vertex, bandwagon radius, bandwagon diameter, bandwagon center, bandwagon periphery are defined. Bounds on bandwagon radius and bandwagon diameter for class of graphs are found. Bandwagon eccentric domination is defined along with bandwagon eccentric domination number $\gamma_{bed}(G)$. Results related to exact values of bandwagon eccentric domination number of class of graphs is obtained.

Competing Interests

Authors have declared that no competing interests exist.

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