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Complex Maxwell's Equations

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Abstract

Maxwell's equations in electromagnetism can be categorized into three distinct groups based on the electromagnetic source when employing quaternions. Each group represents a self-contained system in which Maxwell's equations are applied and validated concurrently, in contrast to the previous approach that did not account for this. It has been noted that the formulation of these Maxwell equations ultimately results in the formulation of Maxwell's equations utilizing the scalar function.

Keywords

Maxwell's Equations, Scalar Function, Proca Equation, Gage Transformation, Quaternion

1. Introduction

Maxwell's equations are fairly well known since they are considered to be the foundation of electromagnetic theory. Maxwell's electromagnetic equations have several specific instances.

Maxwell's equations describe the behavior of magnetic and electric fields. Since the 19th century, when James Clerk Maxwell formulated these equations, their application has proven extremely beneficial to several branches of physics, including electromagnetic, optics, and quantum mechanics. They provide a mathematical model for electrical, optical, and radio technologies, such as power generation, electric motors, wireless communication, lenses, radar, etc. The sources of electric and magnetic fields are defined by the equations as charges, currents, and changes in the fields [1]-[14].

Proca introduced massive photons to Maxwell's theory [15]. The mass term in Maxwell's Lagrangian would break the gauge invariance. The loss of this invariance would weaken the predictive ability of the theory. However, Proca's theory is Lorentz invariant rather than gauge invariant. It is limited to the magnetic and electric fields.

The densities of charges and currents are essential to Maxwell's theory, but not to fields. Maxwell's theory guarantees charge conservation. The continuity equation that links the charge and current densities illustrates this. Maxwell's equations are sometimes expressed in terms of vector and scalar potential fields instead of electric and magnetic potentials. If these potentials are transformed, Maxwell's equations keep invariant, or unchanged under transformation.

It will be unclear whether these potentials are physical phenomena or just abstract mathematical ideas.

Bohm and Aharonov have experimentally demonstrated the existence of these potentials as tangible physical entities [16]. The Lorenz gauge is commonly employed to solve Maxwell's equations using potentials, although various gauges are also accepted. Consequently, the Lorenz gauge must possess a physical component. Quantum electrodynamics arises from the integration of Maxwell's equations and quantum mechanics.

In this context, the interaction between electrons or other charged particles necessitates the involvement of the gauge particle, commonly referred to as the photon. The photon, serving as the gauge boson, possesses a spin of one. While the photon is electrically charged, the electron remains uncharged (neutral). In the standard model of quantum electrodynamics, the photon is considered massless and chargeless. According to the Proca-Maxwell hypothesis, the photon is postulated to have a significant size, yet it lacks any associated charge. However, the Bardeen-Cooper-Schrieffer theory suggests that the phenomenon of superconductivity, which is characterized by the condensation of Cooper pairs, can be coupled to a boson-like state [17]. According to this theory, Cooper pairs are formed when two electrons interact and their spins cancel out, resulting in a spin of zero.

The photon field was linked to the supercurrent, specifically the vector potential, through London's work. This connection resulted in modifications to the gauge, which in turn caused changes to the supercurrent. Given this relationship, it is logical to assume that gauge transformations involve both current density and vector potential. Therefore, it is crucial to approach the analysis of charge-current densities and scalar-vector potentials in a similar manner.

Mathematical structures called quaternions are extensions of complex numbers. They were first used in 1843 by Irish Mathematician Sir William Rowan Hamilton [18]. They are made up of four parts: one scalar component, three vector components, and one matrix component. Quaternions are a mathematical construct that extends the idea of complex numbers to three dimensions and are closed under multiplication. They are especially beneficial for calculations involving rotations in three dimensions.

In mathematics, Quaternions are represented as a four-dimensional vector space with a basis consisting of the real number 1 and three imaginary units, i, j, and k. These unique mathematical entities possess distinct multiplication principles and are utilized for the rotation of three-dimensional objects. Quaternions have found their applications in various fields such as computer graphics, computer vision, robotics, navigation, molecular dynamics, flight dynamics, satellite orbital mechanics, and crystallographic texture analysis. They offer a more compact and efficient representation as compared to rotation matrices and Euler angles. However, comprehending quaternions may not be as straightforward as Euler angles. Despite this, their intriguing features allow for the concise formulation of physical rules. Although quaternions were initially employed by Maxwell in his theory, vectors have now replaced them in the formulation of Maxwell's equations [19] [20] [21] [22] [23].

2. Quaternion Formulation

2.1. Quaternion Continuity Equation

If we have two quaternions $\overleftrightarrow{A} = \left(a_0, \boldsymbol{A}\right)$ and $\overleftrightarrow{B} = \left(b_0, \boldsymbol{B}\right)$, where a_0 and b_0 are scalar parts of the quaternion set. $\boldsymbol{A}$ and $\boldsymbol{B}$ the vector parts of the quaternion. The multiplication rule for two quaternion sets is given by

$$\overleftrightarrow{A}\overleftrightarrow{B} = \left(a_0 b_0 - \boldsymbol{A}\cdot\boldsymbol{B}, a_0\boldsymbol{B} + \boldsymbol{A}b_0 + \boldsymbol{A}\times\boldsymbol{B}\right) \tag{1}$$

The multiplication in (1) consists of scalar part $a_0 b_0$ as a direct product of scalar parts. The second scalar part of multiplication is the scalar product of the vector parts of the quaternion. The vector part of the quaternion product is $a_0\boldsymbol{B} + \boldsymbol{A}b_0 + \boldsymbol{A}\times\boldsymbol{B}$.

By defining the differential operator $\overleftrightarrow{\nabla} = \left(\tilde{\nabla}, \boldsymbol{\nabla}\right)$, the ordinary continuity equation can be transformed into a quaternion continuity equation and the current $\overleftrightarrow{J} = \left(\tilde{J}, \boldsymbol{J}\right)$,

$$\overleftrightarrow{\nabla}\overleftrightarrow{J} = \left(\tilde{\nabla}\tilde{J} - \boldsymbol{\nabla}\cdot\boldsymbol{J}, \tilde{\nabla}\boldsymbol{J} + \boldsymbol{\nabla}\tilde{J} + \boldsymbol{\nabla}\times\boldsymbol{J}\right) \tag{2}$$

We can separate Equation (2) into two equations, one is a vector equation and the other is a scalar equation as,

$$\tilde{\nabla}\tilde{J} - \boldsymbol{\nabla}\cdot\boldsymbol{J} = \wp \tag{3}$$

$$\tilde{\nabla}\boldsymbol{J} + \boldsymbol{\nabla}\tilde{J} + \boldsymbol{\nabla}\times\boldsymbol{J} = \boldsymbol{\wp} \tag{4}$$

But $\overleftrightarrow{\nabla}\overleftrightarrow{J}$ is Lorentz invariant. That means $\overleftrightarrow{\nabla}\overleftrightarrow{J} = 0$.

From above we write two equations,

$$\boldsymbol{\nabla}\times\boldsymbol{J} = -\tilde{\nabla}\boldsymbol{J} - \boldsymbol{\nabla}\tilde{J} \tag{5}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} - \tilde{\nabla}\tilde{J} = 0 \tag{6}$$

If we define, $\overleftrightarrow{\nabla} = \left(-\frac{i}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$ and $\overleftrightarrow{J} = \left(-i\rho c, \boldsymbol{J}\right)$, Equations (5) and (6) give the well-known continuity equations.

2.2. Maxwell Equations

After Maxwell formulated the four laws of electromagnetism, it was observed that there existed a symmetry between the electric field laws and the magnetic field laws. However, this a symmetry was disrupted due to the absence of individual magnetic charges or monopoles. To address this issue, the concept of single magnetic charges was introduced into the physics community. This was done to make Maxwell's equations more symmetrical than their traditional form. With the inclusion of magnetic charges, the magnetic field becomes a non-zero divergence. Furthermore, a current of magnetic charge or monopoles would give rise to a circulating electric field along that current. The admission of magnetic monopoles leads to a particularly symmetric form of Maxwell's equations [24].

Using the definition of quaternions, we can writea set of electric fields with $\left(\tilde{E}, \boldsymbol{E}\right)$, where $\boldsymbol{E}$ the electric field vector and $\tilde{E}$ the field scalar, to write Maxwell equations by defining the following;

$$\boldsymbol{E} = \boldsymbol{\alpha}\tilde{E} = ic\boldsymbol{B}, \quad \boldsymbol{\alpha} = -\frac{i}{c}\boldsymbol{V}, \quad \tilde{E} = i\frac{c\rho}{\sigma}, \quad \sigma = \frac{1}{e} \tag{7}$$

where $\boldsymbol{B}$ the magnetic field, ρ the electric charge density, σ the conductivity of the medium, and e the resistivity of the medium.

We find electric field divergence by employing Equation (7),

$\nabla \cdot \boldsymbol{E} = \boldsymbol{\alpha} \cdot \nabla\tilde{E} + \tilde{E}\nabla \cdot \boldsymbol{\alpha}$

where, $\nabla\tilde{E} = ic\left(e\nabla\rho + \rho\nabla e\right)$, by using Equation (5) we find density gradient.

$\nabla\rho = -\frac{i}{c}\left(\nabla \times \boldsymbol{J}\right) - \frac{1}{c^2}\frac{\partial \boldsymbol{J}}{\partial t}$. Also, we find $\nabla \cdot \boldsymbol{\alpha} = -i\frac{\omega}{c} = -i\frac{\sigma}{c\epsilon_0}$.

where, ω is known as the relaxation time, and it is a measure of how fast a conducting medium reaches electrostatic equilibrium. The permittivity of free space ϵ_0.

The electric field divergence can be expressed using the previous equations as,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} - \frac{e}{c^2}\boldsymbol{V} \cdot \frac{\partial \boldsymbol{J}}{\partial t} + \rho\boldsymbol{V} \cdot \nabla e - i\frac{e}{c}\boldsymbol{V} \cdot \nabla \times \boldsymbol{J} \tag{8}$$

In terms of the electric field, using the relationship of the electric field with the current density, where $e\boldsymbol{j} = \boldsymbol{E}$. This is one way to represent the equation:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} - \frac{1}{c^2}\boldsymbol{V} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \rho\boldsymbol{V} \cdot \nabla e - \frac{i}{c}\boldsymbol{V} \cdot \nabla \times \boldsymbol{E} \tag{9}$$

Equation (9) is generalized Gauss's law in complex form. In terms of magnetic field, it can be expressed as,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} + \rho\boldsymbol{V} \cdot \nabla e - \frac{1}{c^2}\boldsymbol{V} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{V} \cdot \nabla \times \boldsymbol{B} \tag{10}$$

In static cases or perpendicular movements, Equation (10) gives Gauss's law to describe the relationship between a static electric field and the electric charges that cause it.

In vacuum $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$, The Equations (8)-(10) take the following form

$$-\frac{1}{c}\frac{\partial \boldsymbol{F}}{\partial t} - \nabla \Lambda - i\nabla \times \boldsymbol{F} = \mu_0 \acute{J} \tag{11}$$

$$\frac{1}{c^2} = \epsilon_0 \mu_0, \quad -\rho \boldsymbol{V} = \acute{J}, \quad \rho \nabla e = -\nabla \Lambda \tag{12}$$

Looking at the results reached by Arbab [25], we find a similarity between Equations (8)-(10) in vacuum and the equation that describes Maxwell's equations with the Scaler field, which describes extended Maxwell's equations. Λ defines some scalar "magnetic" function representing the fourth component of the electromagnetic 4-vector. This scalar satisfies the wave function. The scalar function is thus a wave moving at the speed of light. Here we find the physical meaning of the scalar function, which is related to the properties of the medium in which the wave propagates. The force acting on electric and magnetic charges is referred to as the generalized Lorentz force. Generally speaking, it is connected to the symmetrized Maxwell's equations.

However, in the current formulation, we did not assume the existence of magnetic charges from the outset. It appears that the electric charge and the magnetic charge are inherently linked.

The bi-quaternion formulation of Maxwell's equation was adopted by Vlaenderen and Waser, who also proposed a scalar function (S) that measures the violation of the Lorenz gauge. They observed that Maxwell's displacement current is mimicked by this scalar field [26].

Also, Proca added a massive photon field to the Maxwell equations to make them more generic [27].

3. Classification of Maxwell's Equations

Three cases representing three different systems will be studied in relation to Equation (10). Previously, Maxwell's equations were used without considering the charges' source, which is represented by the divergence of the electric or magnetic fields. As a result, depending on the type of source, Maxwell's equations will take different forms.

First, Equation (10), which can be divided into two equations, is

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}\left\{1 + \epsilon_0 \boldsymbol{V} \cdot \nabla e\right\} \tag{13}$$

Equation (13) looks like the Proca equation, which describes a massive electromagnetic field. A massive spin-1 field with mass m is described by the Proca equation, a relativistic wave equation, in Minkowski space-time. The massive electromagnetic field is described by the Proca action, which is the equivalent action. The Z and W bosons are two of the three massive vector bosons that are described by the Proca equation and are a part of the Standard Model.

The other is vacuum Ampere's law $\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$.

If we apply the following conditions,

$$\boldsymbol{V}\cdot\nabla e=-\frac{1}{\epsilon_0},\quad \boldsymbol{J}\cdot\nabla e=-\frac{\rho}{\epsilon_0} \tag{14a}$$

We find that, Equation (13) will turn into Maxwell's equation in a vacuum $\nabla\cdot\boldsymbol{E}=0$.

If electric charges are the source of the electric field, as is the case with Maxwell's ordinary equations, we find the second case. $\nabla\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0}$.

In this case, Equation (10) gives Ampere's law $\nabla\times\boldsymbol{B}=\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}+\mu_0\boldsymbol{J}$, Where,

$$\boldsymbol{J}=-\frac{\rho}{\mu_0}\nabla e \quad\text{and}\quad \nabla e=-\mu_0\boldsymbol{V} \tag{14b}$$

The current density in (14b) vanished if $\nabla e=0$. but the divergence ∇e vanished in static case $\boldsymbol{V}=0$, or in the medium exhibits homogeneity in all directions, the gradient of a function is zero at any specific position, indicating that the function remains unchanged at that location. These specific positions are referred to as critical points or stationary points, and they can be classified as maxima, minima, or saddle points based on the second-order partial derivatives of the function. Therefore, we will refer to these conditions as Maxwell's limit, denoted by (14). In the third case, if $\nabla\cdot\boldsymbol{E}=0$ in Equation (10), we get generalized Ampere's law or Proca equation as,

$$\nabla\times\boldsymbol{B}=\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}-\mu_0\boldsymbol{J}\left\{1+\epsilon_0\boldsymbol{V}\cdot\nabla e\right\} \tag{15}$$

The equation of the magnetic field in a vacuum can be obtained by setting the condition in Equation (14a) $\boldsymbol{V}\cdot\nabla e=-\frac{1}{\epsilon_0}$.

After treating the electric field in the previous three cases, we will apply the same conduct to deduce the magnetic field states by multiplying Equation (10) by $-\frac{i}{c}$. We get generalized Gauss's law for magnetism as,

$$\nabla\cdot\boldsymbol{B}=-ic\mu_0\rho\left\{1+\epsilon_0\boldsymbol{V}\cdot\nabla e\right\}-\frac{1}{c^2}\boldsymbol{V}\cdot\frac{\partial\boldsymbol{B}}{\partial t}-\frac{1}{c^2}\boldsymbol{V}\cdot\nabla\times\boldsymbol{E} \tag{16}$$

Using the same treatment as before, we find three cases.

In the first case, Equation (16) can be separated to give,

$$\nabla\cdot\boldsymbol{B}=-ic\mu_0\rho\left\{1+\epsilon_0\boldsymbol{V}\cdot\nabla e\right\} \tag{17}$$

And Faraday's law of induction, $\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$.

Equation (17) mimics the vacuum solution if we apply Condition (14a).

We obtain the second state, which represents the vacuum state, where $\nabla\cdot\boldsymbol{B}=0$, Equation (16) gives generalized Faraday's law of induction as,

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}-ic\mu_0\boldsymbol{J}\left\{1+\epsilon_0\boldsymbol{V}\cdot\nabla e\right\} \tag{18}$$

Equation (18) also transforms to Maxwell's equation by applying Condition (14a).

The third and final case is obtained by setting

$$\nabla \cdot \boldsymbol{B} = -ic\mu_0 \rho \tag{19}$$

Equation (16) with Condition (14) gives,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - ic\mu_0 \boldsymbol{J} \tag{20}$$

Case three is a new set of Proca equations to approach Maxwell's equations.

In Condition (14a) $\boldsymbol{J} \cdot \nabla e = -\frac{\rho}{\epsilon_0}$ we find that, if the gradient vanished everywhere, this will lead to vanishing electric charge density, Equation (19) construes to vacuum Maxwell equation. Also from Condition (14b) $\boldsymbol{J} = -\frac{\rho}{\mu_0} \nabla e$, the current density vanished and Equation (20) is also construed to vacuum Maxwell equation.

From the above, we note that Equations (10) and (16) are generalizations of Maxwell's equations.

According to the above, we can classify Maxwell's equations into three groups.

First group:

Electric Field	Magnetic Field
$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \{1 + \epsilon_0 \boldsymbol{V} \cdot \nabla e\}$	$\nabla \cdot \boldsymbol{B} = -ic\mu_0 \rho \{1 + \epsilon_0 \boldsymbol{V} \cdot \nabla e\}$
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\nabla \times \boldsymbol{B} = \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$

Second group:

Electric Field	Magnetic Field
$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$	$\nabla \cdot \boldsymbol{B} = -ic\mu_0 \rho$
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - ic\mu_0 \boldsymbol{J} \{1 + \epsilon_0 \boldsymbol{V} \cdot \nabla e\}$	$\nabla \times \boldsymbol{B} = \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J}$

Third group:

Electric Field	Magnetic Field
$\nabla \cdot \boldsymbol{E} = 0$	$\nabla \cdot \boldsymbol{B} = 0$
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - ic\mu_0 \boldsymbol{J}$	$\nabla \times \boldsymbol{B} = \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} - \mu_0 \boldsymbol{J} \{1 + \epsilon_0 \boldsymbol{V} \cdot \nabla e\}$

4. Conclusions

Finding symmetry in the equations was done by creating a source for the magnetic field in many published researches, where the equations of the electric field are converted to the magnetic field using some transformations, but it was not considered that the system remains unchanged after using the transformations. Here we found that symmetry of the laws of the two fields is achieved in different physical systems as explained by three groups. Any group of equations that represents a separate physical system in which the complete equations of the set are met. When converting equations using any of the physical methods, such as gauge transformations, this leads to changing the equations in addition to the physical system itself.

The use of quaternions to deduce Maxwell's equations led to their classification into three groups. Each group describes an integrated system that does not intersect with other systems. Therefore, the equations of two different systems should not be used at the same time. Case three is a new set of Proca equations to approach Maxwell's equations. The electromagnetic field to massive photon is described by the Proca equation as the following equations,

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0}-m^2\varnothing\,,\quad \nabla\cdot\boldsymbol{B}=0\,,\quad \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\quad\text{and}\quad \nabla\times\boldsymbol{B}=\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}+\mu_0\boldsymbol{J}-m^2\boldsymbol{A}\,.$$

We can recognize that, the electric field source contains a mass term, which contributes to the formation of the magnetic field. This effect disappears in a vacuum where there is no source of electric field. In contrast with our third group, in a vacuum, there is still an effect of this term appearing in the formation of the electric field as an imaginary mass including a charge density. This imaginary mass can be related to photon mass as proposed by Aquino and Arbab [28] [29].

It is worth noting that the equations of the first and third groups undergo a transformation into Maxwell's equations in a vacuum $\nabla\cdot\boldsymbol{E}=0$, $\nabla\cdot\boldsymbol{B}=0$, $\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$ and $\nabla\times\boldsymbol{B}=\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}$ by applying Condition (14a) $\boldsymbol{V}\cdot\nabla e=-\frac{1}{\epsilon_0}$.

In condition (14a) we notice that the Homogeneity of the medium leads to stability of the value of, *e*, the resistivity of the medium which in turn leads to vanishing the divergence term in the equation of Condition (14) $\boldsymbol{J}\cdot\nabla e=-\frac{\rho}{\epsilon_0}$ we find that, if the gradient vanished everywhere, this will lead to vanishing electric charge density. The real part of the second group is the ordinary Maxwell equations.

And also from Condition (14b) $\boldsymbol{J}=-\frac{\rho}{\mu_0}\nabla e$, the current density vanished. Keep in mind that there is a force acting on the charge that is non-zero in the absence of electric and magnetic fields and their sources, that what Arbab found in his work [29] [30] $\boldsymbol{f}=e\Lambda\boldsymbol{v}$, when the scalar function Λ is constant, the generalized Maxwell's equations are reduced to the ordinary Maxwell's equations.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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