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A Nonlocal Boundary Value Mixed Problem for Second-order Hyperbolic-parabolic Equations in Nonclassical Function Spaces

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Abstract

Aims/ We use an energy method to solve a following boundary-value problem for hyperbolic-parabolic equation with an integral condition :

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial v}{\partial t} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial v}{\partial x}\right) + b(x,t)v = F(x,t) \\ v(x,0) = \psi_1(x),\ 0 \le x \le \ell, \\ \dfrac{\partial v}{\partial t}(x,0) = \psi_2(x),\ 0 \le x \le \ell, \\ u(\ell,t) = E(t),\ 0 \le t \le T, \\ \displaystyle\int_0^\ell xv(x,t)\,dx = H(t),\ 0 \le t \le T. \end{cases}$$

The proof is based on an energy inequality and on the fact that the range of the operator generated is dense.

Keywords: Hyperbolic-Parabolic; A priori Estimate; Nonlocal conditions; Mixed Problem.
2000 Mathematics Subject Classification: 35K20; 35B30; 35D05; 46E40; 46E99.

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1 Introduction

In the present paper, we consider the problem of determining a function $v = v(x,t)$ satisfying, in a weak sense, the linear hyperbolic-parabolic equation

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}\right) + b(x,t)v = F(x,t), \tag{1.1}$$

supplemented by the initial conditions

$$v(x,0) = \psi_1(x),\ 0 \le x \le \ell, \tag{1.2}$$

$$\frac{\partial v}{\partial t}(x,0) = \psi_2(x),\ 0 \le x \le \ell, \tag{1.3}$$

the Dirichlet condition

$$v(\ell,t) = E(t),\ 0 \le t \le T, \tag{1.4}$$

and the integral condition

$$\int_0^\ell xv(x,t)\,dx = H(t),\ 0 \le t \le T, \tag{1.5}$$

where, ψ_1, ψ_2, E and H are sufficiently regular given function, and T and ℓ are the positive constants.

Condition 1.1. *For all* (x,t) *in the closed of* $(0,\ell) \times [0,T]$, *we conditione that*

$$a_0 \le a(x,t) \le a_1,$$
$$a_2 \le \frac{\partial a}{\partial x}(x,t) \le a_3,$$
$$a_4 \le \frac{\partial a}{\partial t}(x,t) \le a_5,$$
$$a_6 \le \frac{\partial^2 a}{\partial x \partial t}(x,t) \le a_7,$$

where $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6$ *and* a_7 *arer positive constants.*

Condition 1.2. *For all* (x,t) *in the closed of* $(0,\ell) \times [0,T]$, *we conditione that*

$$b_0 \le b(x,t) \le a_b,$$
$$b_2 \le \frac{\partial b}{\partial x}(x,t) \le b_3,$$
$$b_4 \le \frac{\partial b}{\partial t}(x,t) \le b_5,$$
$$b_6 \le \frac{\partial^2 b}{\partial x \partial t}(x,t) \le b_7,$$

where $b_0,\ b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6$ *and* b_7 *arer positive constants.*

The data satisfies the following compatibility conditions : for consistency, we have

$$\psi_1(0) = E(0), \quad \text{and} \quad \int_0^\ell x\psi_1(x)\,dx = H(0).$$

The importance of problems with integral conditions has been pointed out by Samarskii [1]. Mathematical modelling by evolution problems with a nonlocal constraint of the form $\int_0^\ell \gamma(x)(x)u(x,t)\,dx = \Gamma(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

See for instance Benouar-Yurchuk [2], Benouar-Bouziani [3]-[4], Bouziani [5]-[8], Cannon et al [10]-[11], Ionkin [12]-[13], Kamynin [14] and Yurchuk [15]-[16]. Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [8]-[17], Byszewski et al [18]-[20], Gasymov [22], Ionkin [12]-[13], Lazhar [24], Mouravey-Philipovski [25], Said-Nadia [26] and Yurtchuk [27].

The results and the method used here are a further elaboration of those in [2].

This basic tool is the energy inequality method which, of course, requires appropriate multipliers and functional spaces.

In this paper, we extend this method to the study of a mixed-type hyperbolic-parabolic equations with Dirichlet and integral conditions.

The remainder of the paper is organized as follows. After this introduction, in section 2, we present some preliminaries and basic definitions. Then in Section 3, we establish a priori estimate and give its several applications. Finally, in section 4, we prove existence of generalized solution.

2 Preliminaries

In point of view of the used method, it is preferable to transform inhomogeneous boundary conditions to homogeneous ones by introducing a new unknown function z defined as follows :

$$z(x,t) = v(x,t) - u(x,t)$$

where

$$u(x,t) = \frac{(-6x^2 + 12\ell x - 5\ell^2)}{\ell^2}E(t) + \frac{(x-\ell)^2}{\ell^2}H(t).$$

Then the problem becomes

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial z}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right) + b(x,t)z = f(x,t) \tag{2.1}$$

$$z(x,0) = \varphi_2(x),\ 0 \le x \le \ell, \tag{2.2}$$

$$\frac{\partial z}{\partial t}(x,0) = \varphi_2(x),\ 0 \le x \le \ell, \tag{2.3}$$

$$z(\ell,t) = 0,\ 0 \le t \le T, \tag{2.4}$$

$$\int_0^\ell xz(x,t)\,dx = 0,\ 0 \le t \le T, \tag{2.5}$$

where

$$f(x,t) = F(x,t) + \frac{(6x^2 - 12\ell x + 5\ell^2)}{\ell^2}(E_1'(t) + E_1'') - \frac{(2x+\ell)(x-\ell)}{\ell}(E_2'(t) + E_2'')(t)$$
$$+ \frac{(6x^2 - 12\ell x + 5\ell^2)b(x,t)}{\ell^2}E(t) + \frac{\partial a}{\partial x}\frac{(x-\ell)^2}{\ell^2}H(t) - \frac{(x-\ell)^2 b(x,t)}{\ell^2}H(t)$$
$$+ \frac{\partial a}{\partial x}\frac{(-6x^2 + 12\ell x - 5\ell^2)}{\ell^2}E(t) - \frac{12(x-\ell)a(x,t)}{\ell^2}E(t) + \frac{2(x-\ell)a(x,t)}{\ell^2}H(t),$$

$$\varphi_1(x) = \psi_1(x) + \frac{(6x^2 - 12\ell x + 5\ell^2)}{\ell^2}E(0) - \frac{(x-\ell)^2}{\ell^2}H(0),$$
$$\varphi_2(x) = \psi_2(x) + \frac{(6x^2 - 12\ell x + 5\ell^2)}{\ell^2}E'(0) - \frac{(x-\ell)^2}{\ell^2}H'(0).$$

Here we conditione that the function φ_1 and φ_2 satisfy conditions of (2.4) and (2.5), that is

$$\varphi_1(0) = 0 \quad \text{and} \quad \int_0^\ell x\varphi_1(x)\,dx = 0.$$

Instead of searching for the function $v(x,t)$, we search for the function z. So the solution of problem (1.1), (1.2), (1.3), (1.4) and (1.5) will be given by $v(x,t) = z(x,t) + u(x,t)$.

The problem (2.1), (2.2), (2.3), (2.4) and (2.5) can be considered as solving the operator equation

$$Az = (f,\ \varphi_1, \varphi_2) = \mathcal{F}, \tag{2.6}$$

where A is an operator defined on E into F. E is the Banach space of functions $L^2(\Omega)$ satisfying the conditions (2.4) and (2.5) with the norm

$$\|z\|_E^2 = \sup_{0\le t\le T}\left\{\int_0^\ell z^2\,dx + \int_0^\ell\left(\frac{\partial z}{\partial x}\right)^2 dx + \int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2 dx\right\},$$

and F is the Hilbert space $L^2(\Omega)\times L^2(0,\ell)\times L^2(0,\ell)$ which consists of elements $\mathcal{F} = (f,\ \varphi_1,\ \varphi_2)$ with the norm

$$\|\mathcal{F}\|_F^2 = \int_0^\ell \varphi_1^2(x)\,dx + \int_0^\ell\left(\frac{d\varphi_1}{dx}\right)^2 dx + \int_0^\ell \varphi_2^2(x)\,dx + \int_\Omega f^2(x,t)\,dxdt.$$

Let $D(A)$ be the set of all functions $u \in L^2(\Omega)$, for $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\partial t} \in L^2(\Omega)$ and satisfying conditions (2.4) and (2.5).

3 Energy Inequality and Its Consequences

Theorem 3.1. *Let conditionption 1.1 be fulfilled. Then the a priori estimate*

$$\|z\|_E \le \lambda\|Az\|_F, \tag{3.1}$$

holds for any function $z \in D(A)$, where λ is a positive constant independent of z.

Proof

Multiplying the equation (2.1) with $Lz = (t-\tau)\int_0^t\int_0^x \frac{\partial z}{\partial t}(\xi,s)\,d\xi ds$ and integrating the results obtained over $\Omega_\tau = (0,\ell)\times(0,T)$. Observe that

$$\int_{\Omega_\tau}(t-\tau)\frac{\partial^2 z}{\partial t^2}\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt + \int_{\Omega_\tau}(t-\tau)\frac{\partial z}{\partial t}\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt$$
$$-\int_{\Omega_\tau}(t-\tau)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt$$
$$+\int_{\Omega_\tau} b(x,t)(t-\tau)z\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt$$
$$=\int_{\Omega_\tau}(t-\tau)f(x,t)\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt. \tag{3.2}$$

Successive integration by parts of integrals on the left-hand of (3.2) are straight-foward but somewhat tedious. We give only their results

$$\int_{\Omega_\tau}(t-\tau)\frac{\partial^2 z}{\partial t^2}\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt = -\int_0^\ell z(x,\tau)\left(\int_0^x z(\xi,\tau)\,d\xi\right)dxdt$$
$$+\int_0^\ell z(x,\tau)\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt + \int_{\Omega_\tau} z\left(\int_0^x \frac{\partial z}{\partial t}\,d\xi\right)dxdt$$
$$-\int_{\Omega_\tau}(t-\tau)\frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial z}{\partial t}\,d\xi\right)dxdt, \tag{3.3}$$

$$\int_{\Omega_\tau}(t-\tau)\frac{\partial z}{\partial t}\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt = -\int_{\Omega_\tau}(t-\tau)z\left(\int_0^x \frac{\partial z}{\partial t}\,d\xi\right)dxdt$$
$$-\int_{\Omega_\tau} z\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt + \int_{\Omega_\tau} z\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt, \tag{3.4}$$

$$-\int_{\Omega_\tau}(t-\tau)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt$$
$$=\int_{\Omega_\tau}(t-\tau)a(x,t)z\frac{\partial z}{\partial x}\,dxdt - \int_{\Omega_\tau}(t-\tau)a(x,t)\varphi_1(x)\frac{\partial z}{\partial x}\,dxdt, \tag{3.5}$$

$$\int_{\Omega_\tau}(t-\tau)f(x,t)\left(\int_0^t\int_0^x \frac{\partial z}{\partial s}(\xi,s)\,d\xi ds\right)dxdt = -\int_{\Omega_\tau}(t-\tau)f(x,t)\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt$$
$$+\int_{\Omega_\tau}(t-\tau)f(x,t)\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt. \tag{3.6}$$

By substituting (3.3), (3.4), (3.5) and (3.6) in (3.2), we obtain

$$\int_{\Omega_\tau}(t-\tau)a(x,t)z\frac{\partial z}{\partial x}\,dxdt - \int_{\Omega_\tau}(t-\tau)a(x,t)\varphi_1(x)\frac{\partial z}{\partial x}\,dxdt$$
$$-\int_{\Omega_\tau}(t-\tau-1)z\left(\int_0^x \frac{\partial z}{\partial t}\,d\xi\right)dxdt - \int_{\Omega_\tau}(t-\tau)\frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial z}{\partial t}\right)dxdt$$
$$-\int_0^\ell z(x,\tau)\left(\int_0^x z(\xi,\tau)\,d\xi\right)dxdt + \int_0^\ell z(x,\tau)\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt$$
$$-\int_{\Omega_\tau} z\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt + \int_{\Omega_\tau} z\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt$$
$$=-\int_{\Omega_\tau}(t-\tau)f(x,t)\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt + \int_{\Omega_\tau}(t-\tau)f(x,t)\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt. \tag{3.7}$$

Multiplying the equation (2.1) with $\dfrac{\partial z}{\partial t}$ and integrating the results obtained over $\Omega_\tau = (0,\ell)\times(0,\tau)$. Observe that

$$\int_{\Omega_\tau} \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial t}\,dxdt + \int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2 dxdt - \int_{\Omega_\tau}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial t}\,dxdt$$
$$+\int_{\Omega_\tau} b(x,t)z\frac{\partial z}{\partial t}\,dxdt = \int_{\Omega_\tau} f(x,t)\frac{\partial z}{\partial t}\,dxdt. \tag{3.8}$$

The standard integration by parts of the terms on the left-hand side of (3.8) leads to

$$\int_{\Omega_\tau} \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell \left(\frac{\partial z}{\partial t}\right)^2 dx - \frac{1}{2}\int_0^\ell \varphi_2^2(x)\,dx, \tag{3.9}$$

$$-\int_{\Omega_\tau}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial t}\,dxdt = \int_0^\tau a(0,t)\frac{\partial z}{\partial t}(0,t)\frac{\partial z(0,t)}{\partial x}\,dt + \frac{1}{2}\int_0^\ell a(x,\tau)\left(\frac{\partial z}{\partial x}\right)^2 dx$$
$$-\frac{1}{2}\int_{\Omega_\tau}\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial z}{\partial x}\right)^2 dxdt - \frac{1}{2}\int_0^\ell a_0(x)\left(\frac{d\varphi_1(x)}{dx}\right)^2 dx - \int_0^\tau a(\ell,t)\frac{\partial z}{\partial t}(\ell,t)\frac{\partial z(\ell,t)}{\partial x}\,dt, \tag{3.10}$$

$$\int_{\Omega_\tau} b(x,t)z\frac{\partial z}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell b(x,\tau)z^2(x,\tau)\,dx - \frac{1}{2}\int_0^\ell b_0(x)\varphi_1^2(x)\,dx - \frac{1}{2}\int_{\Omega_\tau}\frac{\partial b(x,t)}{\partial t}z^2\,dxdt. \tag{3.11}$$

Substituting (3.9), (3.10) and (3.11) into (3.8), we obtain

$$\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2 dxdt + \frac{1}{2}\int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^\ell a(x,\tau)\left(\frac{\partial z}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^\ell b(x,\tau)z^2(x,\tau)\,dx$$
$$+\int_0^\tau a(0,t)\frac{\partial z(0,t)}{\partial t}\frac{\partial z(\ell,t)}{\partial x}\,dt - \int_0^\tau a(\ell,t)\frac{\partial z}{\partial t}(\ell,t)\frac{\partial z(\ell,t)}{\partial x}\,dt = \frac{1}{2}\int_{\Omega_\tau}\frac{\partial b(x,t)}{\partial t}z^2\,dxdt$$
$$+\frac{1}{2}\int_{\Omega_\tau}\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial z}{\partial x}\right)^2 dxdt + \frac{1}{2}\int_0^\ell a_0(x)\left(\frac{d\varphi_1(x)}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^\ell \varphi_2^2(x)\,dx$$
$$+\frac{1}{2}\int_0^\ell b_0(x)\varphi_1^2(x)\,dx + \int_{\Omega_\tau} f(x,t)\frac{\partial z}{\partial t}\,dxdt. \tag{3.12}$$

Combining the equalities (3.7) with (3.12), we get

$$\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2 dxdt+\frac{1}{2}\int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2 dx+\frac{1}{2}\int_0^\ell a(x,\tau)\left(\frac{\partial z}{\partial x}\right)^2 dx+\frac{1}{2}\int_0^\ell b(x,\tau)z^2(x,\tau)\,dx$$
$$=\int_{\Omega_\tau}(\tau-t)a(x,t)z\frac{\partial z}{\partial x}\,dxdt+\int_{\Omega_\tau}(\tau-t)a(x,t)\varphi_1(x)\frac{\partial z}{\partial x}\,dxdt$$
$$+\int_{\Omega_\tau}(\tau+1-t)z\left(\int_0^x\frac{\partial z}{\partial t}\,d\xi\right)dxdt+\int_{\Omega_\tau}(\tau-t)\frac{\partial z}{\partial t}\left(\int_0^x\frac{\partial z}{\partial t}\,d\xi\right)dxdt$$
$$+\int_0^\ell z(x,\tau)\left(\int_0^x z(x,\tau)\,d\xi\right)dxdt-\int_0^\tau a(0,t)\frac{\partial z(0,t)}{\partial t}\frac{\partial z(0,t)}{\partial x}\,dt$$
$$-\int_0^\ell z(x,\tau)\left(\int_0^x\varphi_1(\xi)\,d\xi\right)dxdt+\int_{\Omega_\tau}z\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt-\int_{\Omega_\tau}z\left(\int_0^x\varphi_1(\xi)\,d\xi\right)dxdt$$
$$+\int_{\Omega_\tau}(\tau-t)f(x,\tau)\left(\int_0^x\varphi_1(\xi)\,d\xi\right)dxdt-\int_{\Omega_\tau}(\tau-t)f(x,t)\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt$$
$$+\frac{1}{2}\int_{\Omega_\tau}\frac{\partial b(x,t)}{\partial t}z^2\,dxdt+\frac{1}{2}\int_{\Omega_\tau}\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial z}{\partial x}\right)^2 dxdt+\frac{1}{2}\int_0^\ell a(x,\tau)\left(\frac{d\varphi_1(x)}{\partial x}\right)^2 dx$$
$$+\frac{1}{2}\int_0^\ell b_0(x)\varphi_1^2(x)+\frac{1}{2}\int_0^\ell\varphi_2^2(x)\,dx+\int_{\Omega_\tau}f(x,t)\frac{\partial z}{\partial t}\,dxdt+\int_0^\tau a(\ell,t)\frac{\partial z}{\partial t}(\ell,t)\frac{\partial z(\ell,t)}{\partial x}\,dt, \quad (3.13)$$

In light the Young's inequality and the inequality of Poincare type

$$\int_0^\ell\left(\int_0^x\frac{\partial z}{\partial t}\,d\xi\right)^2 dx\le\frac{\ell}{2}\int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2 dx, \quad (3.14)$$

and the conditionption 1.1 and 1.2, certains terms in the right-hand of (3.13) are then majorized as follows :

$$\int_{\Omega_\tau}(\tau-t)a(x,t)z\frac{\partial z}{\partial x}\,dxdt\le\frac{a_1T}{2}\int_{\Omega_\tau}z^2\,dxdt+\frac{a_1T}{2}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial x}\right)^2 dxdt, \quad (3.15)$$

$$\int_{\Omega_\tau}(\tau-t)a(x,t)\varphi_1(x)\frac{\partial z}{\partial x}\,dxdt\le\frac{a_1T^2}{2}\int_0^\ell\varphi_1^2(x)\,dx+\frac{a_1T}{2}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial x}\right)^2 dxdt, \quad (3.16)$$

$$\int_{\Omega_\tau}(\tau+1-t)z\left(\int_0^x\frac{\partial z}{\partial t}\,d\xi\right)dxdt\le\frac{T+1}{2}\int_{\Omega_\tau}z^2\,dxdt+\frac{(T+1)\ell}{4}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2 dxdt, \quad (3.17)$$

$$\int_{\Omega_\tau}(\tau-t)\frac{\partial z}{\partial t}\left(\int_0^x\frac{\partial z}{\partial t}\right)dxdt\le\frac{T(2+\ell)}{4}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2 dxdt, \quad (3.18)$$

$$\int_0^\ell z(x,\tau)\left(\int_0^x z(x,\tau)\,d\xi\right)dxdt+\int_{\Omega_\tau}z\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt$$
$$\le\frac{(2+\ell)}{4}\int_0^\ell z^2(x,\tau)\,dx+\frac{2+\ell}{8}\int_{\Omega_\tau}z^2\,dxdt, \quad (3.19)$$

$$-\int_0^\ell z(x,\tau)\left(\int_0^x\varphi_1(\xi)\,d\xi\right)dxdt-\int_{\Omega_\tau}z\left(\int_0^x\varphi_1(\xi)\,d\xi\right)dxdt$$
$$\le\frac{\alpha}{2}\int_0^\ell z^2(x,\tau)\,dx+\frac{\ell(1+\alpha T)}{4\alpha}\int_0^\ell\varphi_1^2(x)\,dx+\frac{1}{2}\int_{\Omega_\tau}z^2(x,t)\,dxdt, \quad (3.20)$$

$$\int_{\Omega_\tau}(\tau-t)f(x,t)\left(\int_0^x \varphi_1(\xi)\,d\xi\right)dxdt-\int_{\Omega_\tau}(\tau-t)f(x,t)\left(\int_0^x z(\xi,t)\,d\xi\right)dxdt$$
$$+\int_{\Omega_\tau}f(x,t)\frac{\partial z}{\partial t}\,dxdt\le\frac{1+2T}{2}\int_{\Omega_\tau}f^2(x,t)\,dxdt+\frac{\ell T}{4}\int_0^\ell\varphi_1^2(x)\,dx$$
$$+\frac{1}{2}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2dxdt+\frac{\ell T}{4}\int_{\Omega_\tau}z^2(x,t)\,dxdt. \tag{3.21}$$

Combining the inequalities (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21), choosing $\alpha+3+\ell=b_0$, and using condition 1.1 and 1.2, we obtain

$$\frac{1}{2}\int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2dx+\frac{a_0}{2}\int_0^\ell\left(\frac{\partial z}{\partial x}\right)^2dx+\frac{1}{2}\int_0^\ell z^2(x,\tau)\,dx\le\frac{1}{2}\int_0^\ell\varphi_2^2(x)\,dx+\frac{a_1}{2}\int_0^\ell\left(\frac{d\varphi_1(x)}{dx}\right)^2dx$$
$$+\left[\frac{\ell(1+\alpha T)}{4\alpha}+\frac{\ell T}{4}+\frac{a_1T^2}{2}+\frac{b_1}{2}\right]\int_0^\ell\varphi_1^2(x)\,dx+\left[\frac{(1+a_1)T}{2}+\frac{2+\ell}{8}+\frac{1}{2}+\frac{\ell T}{4}+\frac{b_5}{2}\right]\int_{\Omega_\tau}z^2\,dxdt$$
$$+\left[a_1T+\frac{a_5}{2}\right]\int_{\Omega_\tau}\left(\frac{\partial z}{\partial x}\right)^2dxdt+\frac{2T\ell+\ell+2T}{4}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2dxdt+\frac{1+4T}{8}\int_{\Omega_\tau}f^2(x,t)\,dxdt. \tag{3.22}$$

We also have, by straight foward calculations ,

$$\int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2dx+\int_0^\ell\left(\frac{\partial z}{\partial x}\right)^2dx+\int_0^\ell z^2(x,\tau)\,dx$$
$$\le M_0\left(\int_0^\ell\varphi_2^2(x)\,dx+\int_0^\ell\left(\frac{d\varphi_1(x)}{\partial x}\right)^2dx+\int_0^\ell\varphi_1^2(x)+\int_{\Omega_\tau}f^2(x,t)\,dxdt\right)$$
$$+M_1\left\{\int_{\Omega_\tau}z^2\,dxdt+\int_{\Omega_\tau}\left(\frac{\partial z}{\partial x}\right)^2dxdt+\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2dxdt\right\}, \tag{3.23}$$

where

$$M_0=\frac{\max\left(\frac{1}{2},\quad\frac{\ell(1+\alpha T)}{4\alpha}+\frac{\ell T}{4}+\frac{a_1T^2}{2}+\frac{b_1}{2},\frac{1+4T}{8}\right)}{\min\left(\frac{1}{2},\quad\frac{a_0}{2}\right)}$$

$$M_1=\frac{\max\left(\frac{(1+a_1)T}{2}+\frac{2+\ell}{8}+\frac{1}{2}+\frac{\ell T}{4}+\frac{b_5}{2},\quad\frac{2a_1T+a_5}{2},\quad\frac{2T\ell+\ell+2T}{4}\right)}{\min\left(\frac{1}{2},\quad\frac{a_0}{2}\right)}$$

We eliminate the last term on th right-hand side of (3.23). To do that we use GRONWALL'S Lemma to we obtain

$$\int_0^\ell\left(\frac{\partial z}{\partial t}\right)^2dx+\int_0^\ell\left(\frac{\partial z}{\partial x}\right)^2dx+\int_0^\ell z^2(x,\tau)\,dx\le$$
$$M_0e^{M_1T}\left(\int_0^\ell\varphi_2^2(x)\,dx+\int_0^\ell\left(\frac{d\varphi_1(x)}{\partial x}\right)^2dx+\int_0^\ell\varphi_1^2(x)+\int_{\Omega_\tau}f^2(x,t)\,dxdt\right). \tag{3.24}$$

The right-hand side of (3.24) is independent of τ. By taking the least upper bound of the left side with respect to τ from 0 to T, we get the estimate (3.1) with $\lambda=\sqrt{M_0}e^{\frac{M_1T}{2}}$. This complete the proof of Theorem 3.1. ⊡

Definition 3.1. A solution of the equation

$$\overline{A}z = (f, \varphi_1, \varphi_2) \tag{3.25}$$

is called a strong solution of problem (2.1), (2.2), (2.3), (2.4), (**??**) and (2.5).

Corollary 3.2. *By passing to the limit, the estimate* (3.1) *can be extended to strong solutions, that is we have the inequality*

$$\|z\|_E \leq \lambda \|\overline{A}z\|_F, \ \forall v \in D(\overline{A}). \tag{3.26}$$

From the inequality, we deduce the following statements.

Corollary 3.3. *The range* $R(\overline{A})$ *of the operator* $\overline{A}$ *is closed and* $\overline{R(A)} = R(\overline{A})$.

Corollary 3.4. *A strong solution of the problem* (2.1)*,* (2.2)*,* (2.3)*,* (2.4)*,* (**??**) *and* (2.5) *is unique and depends continuously on* $\mathcal{F} = (f,\ \varphi_1, \varphi_2) \in F$.

Remark 3.1. Hence, to prove that a strong solution of the problem (2.1), (2.2), (2.3), (2.4), (**??**) and (2.5) exists for any element $(f, \varphi_1, \varphi_2) \in F$, it remains to prove that $\overline{R(A)} = F$, where A is an operator define on F by

$$A(z) = \left(\frac{\partial^2 z}{\partial t^2} + \frac{\partial z}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial z}{\partial x} \right),\ \varphi_1,\ \varphi_2 \right).$$

4 Solvability of the Problem

Theorem 4.1. *If for some* $w \in \left\{ z\ :\ \int_\Omega z^2(x,t) < +\infty \right\}$ *and for all element*

$$z \in D_0(A) = \left\{ z \in D(A)\ :\ z(x,0) = \frac{\partial z}{\partial t}(x,0) = 0 \right\},$$

we have

$$\int_0^T \int_0^\ell \frac{\partial^2 z}{\partial t^2} w(x,t)\, dxdt + \int_0^T \int_0^\ell \frac{\partial z}{\partial t} w(x,t)\, dxdt - \int_0^T \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial z}{\partial x} \right) w(x,t)\, dxdt = 0, \tag{4.1}$$

then w *vanishes almost everywhere in* Ω.

Proof

Replacing in (4.1), w by $w = -\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi, t)\, d\xi + \frac{\partial z}{\partial t}$, it follows that

$$\begin{aligned} &\int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2} \left(-\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi,t)\, d\xi + \frac{\partial z}{\partial t} \right) dxdt \\ &+ \int_0^\tau \int_0^\ell \frac{\partial z}{\partial t} \left(-\frac{\partial^3 z}{\partial t^2} - \int_0^x \frac{\partial z}{\partial t}(\xi,t)\, d\xi + \frac{\partial z}{\partial t} \right) dxdt \\ &- \int_0^\tau \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial z}{\partial x} \right) \left(-\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi,t)\, d\xi + \frac{\partial z}{\partial t} \right) dxdt = 0. \end{aligned} \tag{4.2}$$

Integrating by parts and taking into account (2.5) each term of (4.2), we obtain

$$-\int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2} \frac{\partial^3 z}{\partial t^3}\, dxdt = -\frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x,\tau)}{\partial t^2} \right)^2 dx + \frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x,0)}{\partial t^2} \right)^2 dx, \tag{4.3}$$

$$\int_0^\tau \int_0^\ell \frac{\partial z}{\partial t}\frac{\partial z}{\partial t}\,dxdt = \int_0^\tau \int_0^\ell \left(\frac{\partial z}{\partial t}\right)^2 dxdt, \tag{4.4}$$

$$\int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell \left(\frac{\partial z(x,\tau)}{\partial t}\right)^2 dx, \tag{4.5}$$

$$-\int_0^\tau \int_0^\ell \frac{\partial z}{\partial t}\frac{\partial^3 z}{\partial t^3}\,dxdt = \int_0^\tau \int_0^\ell \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt - \int_0^\ell \frac{\partial z(x,\tau)}{\partial t}\frac{\partial^2 z(x,\tau)}{\partial t^2}\,dx, \tag{4.6}$$

$$\begin{aligned}-\int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2}\left(\int_0^x \frac{\partial z}{\partial t}(\xi,t)\,d\xi\right)dxdt = &-\int_0^\ell \frac{\partial z(x,\tau)}{\partial t}\left(\int_0^x \frac{\partial z(\xi,\tau)}{\partial t}\,d\xi\right)dx\\ &+\int_{\Omega_\tau} \frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial^2 z}{\partial t^2}\,d\xi\right)dxdt,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\int_0^\tau \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\frac{\partial^3 z}{\partial t^3}\,dxdt = &\int_0^\ell a(x,\tau)\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial t^2}\,dx + \int_0^\ell \frac{\partial a(x,\tau)}{\partial x}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t^2}\,dx\\ &-\int_0^\ell \int_0^\tau a(x,t)\frac{\partial^2 z}{\partial t^2}\frac{\partial^3 z}{\partial t\partial x^2}\,dxt - \int_0^\ell \int_0^\tau \frac{\partial^2 a}{\partial x\partial t}\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial x}\,dxdt,\end{aligned} \tag{4.8}$$

$$-\int_0^\tau \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial t}\,dxdt = -\int_0^\tau \int_0^\ell a(x,t)\frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial t}\,dxdt - \int_0^\tau \int_0^\ell \frac{\partial a(x,t)}{\partial x}\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\,dxdt, \tag{4.9}$$

$$\int_0^\tau \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\left(\int_0^x \frac{\partial z}{\partial t}(\xi,t)\,d\xi\right)dxdt = -\int_0^\ell \int_0^\tau a(x,t)\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}\,dxdt. \tag{4.10}$$

Substituting the equalities (4.3) (4.4), (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) into (4.2), we obtain

$$\begin{aligned}&-\frac{1}{2}\int_0^\ell \left(\frac{\partial^2 z(x,\tau)}{\partial t^2}\right)^2 dx + \frac{1}{2}\int_0^\ell \left(\frac{\partial^2 z(x,0)}{\partial t^2}\right)^2 dx + \int_0^\tau \int_0^\ell \left(\frac{\partial z}{\partial t}\right)^2 dxdt\\ &+\frac{1}{2}\int_0^\ell \left(\frac{\partial z(x,\tau)}{\partial t}\right)^2 dx + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt - \int_0^\ell \frac{\partial z(x,\tau)}{\partial t}\frac{\partial^2 z(x,\tau)}{\partial t^2}\,dx\\ &-\int_0^\ell \frac{\partial z(x,\tau)}{\partial t}\left(\int_0^x \frac{\partial z(\xi,\tau)}{\partial t}\,d\xi\right)dx + \int_0^\ell \int_0^\tau \frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial^2 z}{\partial t^2}\,d\xi\right)dxdt\\ &-\int_0^\tau \int_0^\ell \frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial z}{\partial t}(\xi,t)\,d\xi\right)dxdt - \int_0^\ell \int_0^\tau a(x,t)\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}\,dxdt\\ &-\int_0^\ell \int_0^\tau \frac{\partial^2 a}{\partial x\partial t}\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial x}\,dxdt - \int_0^\tau \int_0^\ell a(x,t)\frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial t}\,dxdt - \int_0^\tau \int_0^\ell \frac{\partial a(x,t)}{\partial x}\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\,dxdt\\ &+\int_0^\ell a(x,\tau)\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial t^2}\,dx + \int_0^\ell \frac{\partial a(x,\tau)}{\partial x}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t^2}\,dx - \int_0^\ell \int_0^\tau a(x,t)\frac{\partial^2 z}{\partial t^2}\frac{\partial^3 z}{\partial t\partial x^2}\,dxdt = 0.\end{aligned} \tag{4.11}$$

The application of Young's inequality and (3.14) to the terms of the above equality gives

$$-\int_0^\ell \frac{\partial z(x,\tau)}{\partial t}\frac{\partial^2 z(x,\tau)}{\partial t^2}\,dx \le \frac{1}{2}\int_0^\ell \left(\frac{\partial z(x,\tau)}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^\ell \left(\frac{\partial^2 z(x,\tau)}{\partial t^2}\right)^2 dx, \tag{4.12}$$

$$-\int_0^\ell \frac{\partial z(x,\tau)}{\partial t}\left(\int_0^x \frac{\partial z(\xi,\tau)}{\partial t}\,d\xi\right)dx \le \frac{8+\ell}{8}\int_0^\ell \left(\frac{\partial z(x,\tau)}{\partial t}\right)^2 dx, \tag{4.13}$$

$$\int_{\Omega_\tau} \frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial^2 z}{\partial t^2}\right) dxdt \le \frac{1}{2}\int_{\Omega_\tau}\left(\frac{\partial z}{\partial t}\right)^2 dxdt + \frac{\ell}{4}\int_{\Omega_\tau}\left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt, \tag{4.14}$$

$$-\int_0^\tau\int_0^\ell \frac{\partial z}{\partial t}\left(\int_0^x \frac{\partial z}{\partial t}(\xi,t)\,d\xi\right) dxdt \le \frac{2+\ell}{4}\int_0^\ell\int_0^\ell \left(\frac{\partial z}{\partial t}\right)^2 dx, \tag{4.15}$$

$$\begin{aligned}
&-\int_0^\ell\int_0^\tau a(x,t)\frac{\partial z}{\partial t}\frac{\partial z}{\partial x}\,dxdt - \int_0^\tau\int_0^\ell \frac{\partial a(x,t)}{\partial x}\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\,dxdt \\
&\le \frac{a_1+a_3\varepsilon}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial z}{\partial t}\right)^2 dxdt + \frac{a_1\varepsilon+a_3}{2\varepsilon}\int_0^\ell\int_0^\tau\left(\frac{\partial z}{\partial x}\right)^2 dxdt,
\end{aligned} \tag{4.16}$$

$$-\int_0^\ell\int_0^\tau \frac{\partial^2 a}{\partial x\partial t}\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial x}\,dxdt \le \frac{a_7}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial z}{\partial x}\right)^2 dxdt + \frac{a_7}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt, \tag{4.17}$$

$$\int_0^\ell \frac{\partial a(x,\tau)}{\partial x}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t^2}\,dx \le \frac{a_3}{2}\int_0^\ell\left(\frac{\partial z(x,\tau)}{\partial x}\right)^2 dx + \frac{a_3}{2}\int_0^\ell\left(\frac{\partial^2 z(x,\tau)}{\partial t^2}\right)^2 dx, \tag{4.18}$$

$$\int_0^\ell a(x,\tau)\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial t^2}\,dx \le \frac{a_1}{2}\int_0^\ell\left(\frac{\partial^2 z}{\partial x^2}\right)^2 dx + \frac{a_1}{2}\int_0^\ell\left(\frac{\partial^2 z}{\partial t^2}\right)^2 dx, \tag{4.19}$$

$$-\int_0^\ell\int_0^\tau a(x,t)\frac{\partial^2 z}{\partial t^2}\frac{\partial^3 z}{\partial t\partial x^2}\,dxdt \le \frac{a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt + \frac{a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial^3 z}{\partial t\partial x^2}\right)^2 dxdt. \tag{4.20}$$

$$-\int_0^\tau\int_0^\ell a(x,t)\frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial t}\,dxdt \le \frac{a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial^2 z}{\partial x^2}\right)^2 dxdt + \frac{a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial z}{\partial t}\right)^2 dxdt, \tag{4.21}$$

If we substitute (4.12), (4.13), (4.14), (4.14), (4.15), (4.16), (4.17), (4.18), (4.19), (4.20) and (4.21) into (4.11), we obtain

$$\begin{aligned}
&-\frac{a_3}{2}\int_0^\ell\left(\frac{\partial z(x,\tau)}{\partial x}\right)^2 dx - \frac{a_1+1}{2}\int_0^\ell\left(\frac{\partial^2 z(x,\tau)}{\partial x^2}\right)^2 dx - T\int_0^\ell\left(\frac{\partial^3 z}{\partial t\partial x^2}\right)^2 dx \\
&-\frac{\ell}{8}\int_0^\ell\left(\frac{\partial z(x,\tau)}{\partial t}\right)^2 dx - \frac{a_3+a_1\varepsilon-1}{2}\int_0^\ell\left(\frac{\partial^2 z}{\partial t^2}\right)^2 dx \\
&\le \frac{(2+a_7)\varepsilon+a_1\varepsilon+a_3}{2\varepsilon}\int_0^\ell\int_0^\tau\left(\frac{\partial z}{\partial x}\right)^2 dxdt + \frac{2+a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial^2 z}{\partial x^2}\right)^2 dxdt \\
&+\frac{3+\ell 2a_1+2a_3\varepsilon+2a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial z}{\partial t}\right)^2 dxdt \\
&+\frac{\ell+2a_7}{4}\int_0^\ell\int_0^\tau\left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt + \frac{2+a_1}{2}\int_0^\ell\int_0^\tau\left(\frac{\partial^3 z}{\partial t\partial x^2}\right)^2 dxdt.
\end{aligned} \tag{4.22}$$

Hence, if $\varepsilon > 0$, and $a_3 + \varepsilon a_1 - 1 = 1$, then (4.26) implies that

$$-\int_0^\ell \left(\frac{\partial z(x,\tau)}{\partial x}\right)^2 dx - \int_0^\ell \left(\frac{\partial^2 z(x,\tau)}{\partial x^2}\right)^2 dx - \int_0^\ell \left(\frac{\partial^3 z}{\partial t \partial x^2}\right)^2 dx - \int_0^\ell \left(\frac{\partial z(x,\tau)}{\partial t}\right)^2 dx$$
$$-\int_0^\ell \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dx \le \beta \left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t}\right)^2 dxdt\right.$$
$$\left. + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2}\right)^2 dxdt\right]. \tag{4.23}$$

Where

$$\beta = \frac{\beta_2}{\beta_1},$$

$$\beta_1 = \min\left(\frac{a_3}{2},\ \frac{1+a_1}{2},\ T,\ \frac{\ell}{4},\ \frac{1}{2}\right),$$

$$\beta_2 = max\left(\frac{(2+a_7)(2+a_3)+2a_1}{2+a_3},\ \frac{\ell+2a_7}{4},\ \frac{2+a_1}{2}\right).$$

It follows from (4.27),

$$-\frac{\partial}{\partial \tau}\left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt\right.$$
$$\left. + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2}\right)^2 dxdt\right] \le \beta \left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2}\right)^2 dxdt\right.$$
$$\left. + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t}\right)^2 dxdt. + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2}\right)^2 dxdt\right]. \tag{4.24}$$

An integration of (4.28) over $[0, \tau]$ gives

$$\left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t}\right)^2 dxdt\right.$$
$$\left. + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2}\right)^2 dxdt + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2}\right)^2 dxdt\right] e^{\beta\tau}, \le 0 \quad \forall \tau \in [0, T]. \tag{4.25}$$

It follows the above inequality (4.21) that $w = 0$ almost everywhere on the Domaine $(0, \ell) \times [0\tau]$. The length τ does not depend on the origin, so we can proceed in the same way a finite number of times to show that $w = 0$ in $0, \ell) \times [0, T]$. Thus, the proof of Theorem (4.1) is completed. ⊡

Theorem 4.2. *Under th conditionptions of the Theorem 4.1, the range $R(A)$ of the operator is dense in F.*

Proof

Since F is a Hilbert space, $R(A)$ is dense in F is equivalent to the orthogonality of the vecteur $\eta = (\eta_0, \eta_1, \eta_2) \in F$ to the set $R(A)$, that is, if and only if the relation

$$\int_0^T \int_0^\ell \frac{\partial^2 z}{\partial t^2}\eta_0(x,t)\,dxdt + \int_0^T \int_0^\ell \frac{\partial z}{\partial t}\eta_0(x,t)\,dxdt - \int_0^T \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z}{\partial x}\right)\eta_0(x,t)\,dxdt$$
$$+ \int_0^\ell \varphi_1(x)\eta_1(x)\,dx + \int_0^\ell \varphi_2(x)\eta_2(x)\,dx = 0, \tag{4.26}$$

where z runs over E and $\eta = (\eta_0, \eta_1, \eta_2) \in F$, implies $\eta = 0$.

Putting $z \in D_0(A)$ in (4.26), we obtain

$$\int_0^T \int_0^\ell \frac{\partial^2 z}{\partial t^2}\eta_0(x,t)\,dxdt + \int_0^T \int_0^\ell \frac{\partial z}{\partial t}\eta_0(x,t)\,dxdt - \int_0^T \int_0^\ell \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}\right)\eta_0(x,t)\,dxdt. \quad (4.27)$$

Therefore, the Theorem(4.1) implies that $\eta_0 = 0$. Thus (4.26) becomes

$$\int_0^\ell \varphi_1(x)\eta_1(x)\,dx + \int_0^\ell \varphi_2(x)\eta_2(x)\,dx = 0. \quad (4.28)$$

Since the operators $z(x,t) \longmapsto \varphi_1(x)$ and $z(x,t) \longmapsto \frac{\partial z}{\partial t}(x,0) = \varphi_2(x)$ are dense in $L^2(0,\ell)$, the last relation (4.28) gives $\eta_1 = \eta_2 = 0$. Hence $\eta \equiv 0$, and thus the cloosed $\overline{R(A)}$ of $R(A)$ is equal to F. This completed the proof of Theorem 4.2. $\boxdot$

5 Conclusion

We use the a priori estimates method to prove the existence, uniqueness and the continuons dependence, upon the data, of solutions to new mixed problem type, boundary-value problem for hyperbolic-parabolic equation with an integral condition.

This results contributes to the development of the a priori estimates method for solving such problems. The question related to these problems are so miscellameous that the elaboration of a general theory is still premature. Therefore, the investigation of these problems requires at every time a separate study.

Competing Interests

The authors declare that no competing interests exist.

References

[1] A. A. Samarkii; Some problems in differential equtions theory, *Differents. Uravn.*, 16:11 (1980), p.1925 - 1935.

[2] N. E. Benouar and N. I. Yurchuk; Mixed problem with an integral condition for parabolique equation with the Bessel operator, *Differentsial'nye Uravneniya,* 27 (1991), p. 2094 - 2098.

[3] N. E Benouar and A.Bouziani; Problème mixte avec conditions intégrales pour une classe d'équation parabolique, *C.R.A.S, Paris, 321, Série I,* (1995), p. 1177-1182.

[4] N. E Benouar and A.Bouziani; Mixed problem with integral conditions for third order parabolic equation, *Kobe J. Math.*, 15 (1998), p. 47-58.

[5] A. Bouziani; Problème mixte avec conditions intégrales pour une classe d'équations paraboliques. *C. R. Acad. Sci.* Paris 321. serie 1. (1995), p. 1177-1182.

[6] A.Bouziani; Mixed problem with boundary integral conditions for a certain parabolic equations, *Journal of Applied Mathematics and Stochastic Analysis*, 9:3 (1996), p. 323 - 330.

[7] A. Bouziani; Solution forte d'un problème mixte avec une condition non locale pour une classe d'équations paraboliques, *Maghreb Math. Rev.*, (1997), p. 1-17.

[8] A. Bouziani; Sur un problème parabolique avec les conditions intégrales, *Annale de Mathématiques*, 6(1999), p. 55 - 66.

[9] A. Bouziani; On the solvability of nonlocal pluriparabolic problem, *Electronic Journal of differential Equations*, vol. 2001 (2001), No. 21, p. 1-16.

[10] J. R. Cannon; *The one-dimensional heat equation,*in Encyclopedia of mathematic and its applications, 23, Addison-Wesley, Reading, MA, Menlo Park, CA, 1984.

[11] J. R. Cannon and J. van der Hoek; The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, *Boll. Uni. Math. Ital. Suppl.*, 1 (1981), p. 253 - 282.

[12] N. I. Ionkin; Solution of boundary value problems in heat conduction theory with with nonlocal boundary conditions, *Differents. Uravn.*, 13:2 (1977), p. 294 - 304.

[13] N. I. Ionkin and E. I. Moiceev; Solution of boundary value problem in heat conduction theory with nonlocal boundary conditions, *Differents. Uravn.*, 13 (1977), p. 294 - 304.

[14] N. I. Kamynin; A boundary value problem in the theory of the heat conduction with nonclassical boundary condition, *Th., Vychisl., Mat., Fiz.*, 4:6, (1964), p. 1006 - 1024.

[15] N. I. Yurchuk; Mixed problem with an integral condition for certain parabolic equations, *Differents. Uravn.*, 22:12, (1986), p. 2117 - 2126.

[16] Djibibe Moussa Zakari et al. Continuous dependence of solutions to mixed boundary value problems for a parabolic equation, *Electronic Journal of Differential Equations*, Vol. 2008(2008), No. 17, p. 1-10.

[17] A. Bouziani; On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in nonclassical function spaces. *IJMMS.*, 30:7 (2002), p. 435 - 447.

[18] L. Byszewski; Existence and uniqueness of solutions of nonlocal problem for hyperboic equation, *J. Appl. Math. Stochastic. Anal.,*3 (1990), p. 163-168.

[19] L. Byszewski; Theorem about existence and uniqueness of continuous solutions for non local problem for nonlinear hyperbolic equation, *Applicable Analysis*, 40(1991), p. 173-180.

[20] L. Byszewski; Differential and functional-differential problems together with nonlocal conditions, *Cracow University of technology Monograph,* 184, Cracow 1995.

[21] J. R. Cannon; The solution of the heat equation subject to the specification of energy, Quart. Appl. Math. 21:2 (1963), p. 155 - 160.

[22] E. A. Gasymov; Mixed problems on the conjugaison of parbolic system of different order with nonlocal boundary conditions, *Differentsial'nye Uravnenya*, 26, No 8, (1991), p. 1003 - 1012.

[23] A. V. Kartynnik; Problème mixte à trois points avec conditions intégrales suivant les variables spaciales pour les équations paraboliques du second ordre" Equations différentielles, 26:9 (1990), p. 1568 - 1775.

[24] B. Lazhar; Parabolic equations with nonlocal conditions, *Applied Mathematical Sciences*, Vol. 1,(2007), No. 21, p. 1041 - 1048.

[25] L. A. Mouravey and V. Philipovski; Sur un problème avec une condition aux limites non locale pour une équation parabolique, *(Russian) Mat. Sb.*, 182 (1991), No.10, p.1479 - 1512.

[26] M. Said and L. Nadia; On a nonlocal singular mixed evolution problem, 53 (2001), p. 79 - 89.

[27] N. I. Yurchuk; Solvability of boundary value problems for certain operator-differential equation, *Differentsial'nye Uravney*, 13, (1977), No. 4, p. 626 - 636.

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