British Journal of Mathematics & Computer Science
4(5): 590-596, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

# On the Gaussıan and Mean Curvatures of Parallel Hypersurfaces in $E_1^{n+1}$

**Ayşe Yaşar Yavuz[1], F. Nejat Ekmekci[2] and Yusuf Yaylı[2]**

*[1]Necmettin Erbakan University, Education of Mathematics, Konya, Turkey.*
*[2]Ankara University, Faculty of Sciences, Department of Mathematics, Ankara, Turkey.*

***Original Research Article***

*Received: 12 July 2013*
*Accepted: 05 November 2013*
*Published: 25 November 2013*

## Abstract

Let M be a hypersurfaces in (n+1) dimensional Lorentzian space $E_1^{n+1}$ and $\bar{M}$ be a parallel hypersurfaces to M. Before now in [2] the theorem was proved on M in Euclidean space, but now we prove this theorem on $\bar{M}$ in Lorentzian Space. In this study, we give higher order Gaussian curvatures of $\bar{M}$ in Lorentzian space by using its principal curvatures and we proved the theorem with induction method by using higher order Gaussian curvatures of $\bar{M}$ in Lorentzian space.

Keywords: Gaussian curvatures, mean curvatures, parallel hypersurfaces, higher order Gaussian curvatures.

## 1 Introduction

For an integer $v$ with $0 \le v \le n,$ changing the first $v$ plus signs above to minus gives a metric tensor

$$< v_p, w_p >= -\sum_{i=1}^{v} v_i w_i + \sum_{j=v+1}^{n} v_i w_i$$

of index $v$ .The resulting semi-Euclidean space $R_v^n$ reduces to $R^n$ if $v = 0$. For $n \ge 2$, $R_1^n$ is called Minkowski $n - space$;if $n$= 4 it is the simplest example of a relativistic spacetime.

The common value $v$ of index $g_p$ on a semi-Riemannian manifold $M$ is a called the index of $0 \le v \le n = dimM$ . If $v = 0$, $M$ is $Riemannian\, manifold$; each $g_p$ is then a (positive definite) inner product on $T_p(M)$. If $v = 1\, and\, n \ge 2$, $M$ is a Lorentz manifold. Fix the notation

**Corresponding author: ayasar@konya.edu.tr;*

$$\varepsilon_i = \begin{cases} -1 & for\ 1 \le i \le v, \\ +1 & for\ v+1 \le i \le n \end{cases}.$$

Then the metric of $R_v^n$ can be written

$$g = \sum \varepsilon_i du_i \otimes du_i.$$

A tangent vector $v$ to $M$ is

$$\begin{gathered} spacelike\ if\ <v,v>>0\ or\ <v,v>=0, \\ null\ if\ <v,v>=0\ and\ v \neq 0 \\ timelike\ if\ <v,v><0 \quad [1] \end{gathered}$$

Normal curvatures, principal curvatures for hypersurfaces and the relevant higher order Gaussian curvatures are invariants independent of the choice of coordinates. There has been some recent studies on the relations between higher order Gaussian curvatures of a hypersurfaces $M$in $E_1^{n+1}$ and that another hypersurfaces $\bar{M}$ which is parallel to $M$. These invariants relevant definitions and theorems are generalized by replacing $E_1^{n+1}$ with a $(n+1)$ −Lorentzian manifold.

Let $M$ and $\bar{M}$ are two hypersurfaces in $E_1^{n+1}$ with unit normal vector $N$of $M$.

$$N = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial_{x_i}}$$

where each $\alpha_i$ is a $C^\infty$function of $M$. If there is a function f, from $M$ to $\bar{M}$ such that

$$\begin{aligned} &f: M \to \bar{M} \\ &P \to f(P) = P + rN_P \end{aligned}$$

then $\bar{M}$ is called parallel hypersurfaces of $M$, where $r \in R$.[1]

$X \in \chi(M) and\ \bar{X} \in \chi(\bar{M})$ vector fields and we have

$$X = \sum_{i=1}^{n+1} b_i \frac{\partial}{\partial_{x_i}}$$

$$\bar{X} = \sum_{i=1}^{n+1} \bar{b}_i \frac{\partial}{\partial_{x_i}}$$

$$P \in M,\ \ b_i(P) = \bar{b}_i\big(f(P)\big),\ 1 \le i \le n,$$

$i) f_*(X) = \bar{X} + rS(X)$

$ii) \bar{S}\big(f_*(X)\big) = S(X)$

$iii)$ If $k$ is principal curvatures of $M$ at the point $P$ in direction $X$ , then $\frac{k}{1+rk}$ is the principal curvatures of $\bar{M}$ at the point $f(P)$ in direction $f_*(X)$, that is, $\bar{S}\big(f_*(X)\big) = \frac{k_1}{1+rk_1} f_*(X)$ which means that $f$ preserves principal directions, where $f_*$ is the differential of $f$ and we know that [3]

$$\bar{S}\big(f_*(X_1)\big) = \frac{k_1}{1+rk_1} f_*(X_1)$$
$$\bar{S}\big(f_*(X_2)\big) = \frac{k_2}{1+rk_2} f_*(X_2)$$
$$\vdots$$
$$\bar{S}\big(f_*(X_n)\big) = \frac{k_n}{1+rk_n} f_*(X_n).$$

## 2. Basic Consepts

### Definition 1:

Let M and $\bar{M}$ are two hypersurfaces in $E_1^{n+1}$ with unit normal vector N of M.

$$N = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial_{x_i}}$$

where each $\alpha_i$ is a $C^\infty$function of M. If there is a function f from M to $\bar{M}$ such that

$$f: M \to \bar{M}$$
$$P \to f(P) = P + rN_P$$

then $\bar{M}$ is called parallel hypersurfaces of M, where r∈R.

### Definition 2:

Let $N$ be a unit normal vector field on a semi-Riemannian hypersurfaces $\bar{M}$ . The tensor field $S$ on $M$ such that

$$\langle S(V), W\rangle = \langle II(V,W), N\rangle \quad for\ all\ V, W \in \aleph(M)$$

is called shape operator $M$ derived from $N$.

As usual, $S$ determines a linear operator $S_P: T_M(P) \to T_M(P)$ at each point $p \in M$. [1]

### Definition 3:

Let $M$ be a hypersurfaces in $E_1^{n+1}$and $S$ denotes the shape operator on M, at $P \in M$. The function $H$ defined by

$$H: M \to R$$
$$P \to H(P) = Trace\ S(P)$$

is called the mean curvature function of $M$ and the real number $H(P)$ is called mean curvature of $M$ at the point $P$. [3]

## Definition 4:

Let $M$ be a hypersurfaces in $E_1^{n+1}$ and $S$ denotes the shape operator on M, at $P \in M$. The function $K$ defined by

$$K: M \to R$$
$$P \to K(P) = \varepsilon det\, S(P)$$

is called the Gaussian curvature function of $M$ and the real number $K(P)$ is called Gaussian curvature of $M$ at the point $P$. [3]

## Definition 5:

Let M be a hypersurfaces in $E_1^{n+1}$ and $T_M(P)$ be a tangent space on M, at P∈M. If $S_P$ denotes the shape operator on M, then

$$S_P: T_M(P) \to T_M(P)$$

is a linear mapping. If we denote the characteristic vectors by $k_1$, $k_2$, ... ,$k_n$ and the corresponding characteristic vectors by $x_1$, $x_2$ ,..., $x_n$ of $S_P$ then $k_1, k_2, \ldots, k_n$ are the principal curvatures and $x_1$, $x_2$, ..., $x_n$ are the principal directions of M, at P∈M . On the other hand, if we use the notions $\varepsilon_i$=+1

$$K_1^{(n)}(k_1, k_2, \ldots, k_n) = \varepsilon_1 k_1 + \sum_{i=2}^{n} \varepsilon_i k_i$$
$$K_2^{(n)}(k_1, k_2, \ldots, k_n) = \sum_{i=1<j}^{n} \varepsilon_i k_i k_j + \sum_{i\neq 1<j}^{n} \varepsilon_i k_i k_j$$
$$K_3^{(n)}(k_1, k_2, \ldots, k_n) = \sum_{i=1<j<t}^{n} \varepsilon_i k_i k_j k_t + \sum_{i\neq 1<j<t}^{n} \varepsilon_i k_i k_j k_t$$
$$\vdots$$
$$K_n^{(n)}(k^1, k^2, \ldots, k_n) = \varepsilon_1 \prod_{i=1}^{n} k_i$$

then the characteristic polynomial of S(P) becomes

$$P_{S(P)}(k) = k^n + (-1)K_1^{(n)}k^{n-1} + \ldots + (-1)^n K_n^{(n)}$$

and $K_1$,$K_2$, ... , $K_n$ are uniquely determined , where the functions $K_i$ are called the higher ordered Gaussian curvatures of the hypersurface $M$.

$\bar{M}$ be a parallel hypersurfaces to $M$ in $E_1^{n+1}$. If $\bar{k_\imath}$ denotes the i-th principal curvature function on $\bar{M}$, then, in a similar way, we can write the higher order Gaussian curvatures on $\bar{M}$as follows;

$$\overline{K}_1^{(n)}(\overline{k_1}, \overline{k_2}, \dots, \overline{k_n}) = \varepsilon_1 \overline{k}_1 + \sum_{i=2}^{n} \varepsilon_i \overline{k_\iota}$$

$$\overline{K}_2^{(n)}(\overline{k}_1, \overline{k}_2, \dots, \overline{k}_n) = \sum_{i=1<j}^{n} \varepsilon_i \overline{k}_i \overline{k}_j + \sum_{i\neq 1<j}^{n} \varepsilon_i \overline{k}_i \overline{k}_j$$

$$\overline{K}_3^{(n)}(\overline{k}_1, \overline{k}_2, \dots, \overline{k}_n) = \sum_{i=1<j<t}^{n} \varepsilon_i \overline{k}_i \overline{k_j} \overline{k}_t + \sum_{i\neq 1<j<t}^{n} \varepsilon_i \overline{k}_i \overline{k}_j \overline{k_t}$$

$$\vdots$$

$$\overline{K}_n^{(n)}(\overline{k}_1, \overline{k}_2, \dots, \overline{k}_n) = \varepsilon_1 \prod_{i=1}^{n} \overline{k_\iota}$$

where $\overline{k_\iota} = \frac{k_i}{1++rk_i}, 1 \leq i \leq n.$

## Theorem:

Let $\overline{M}$ be a parallel hypersurfaces in $E_1^{n+1}$ and $\overline{K_1}, \overline{K_2}, \dots, \overline{K_n}$ are called the higher order Gaussian curvatures and $\overline{k}_1, \overline{k}_2, \dots, \overline{k}_n$ are the principal curvatures at the point $f(P) \in \overline{M}$. Let us define a function

$$\varepsilon_i = +1, \varepsilon_1 = \pm 1 \ (i \neq 1)$$

$$\varphi : \overline{M} \to R$$

$$P \to \varphi(P) = \varphi(r, \overline{k}_1, \overline{k}_2, \dots, \overline{k}_n) = \prod_{i=1}^{n} (1 + \varepsilon_i r \overline{k_\iota})$$

such that $\varphi$ function is

$$\varphi(r, \overline{k}_1, \overline{k}_2, \dots, \overline{k}_n) = r\overline{K_1} + r^2 \overline{K_2} + \cdots + r^n \overline{K_n}.$$

where $\overline{k_\iota} = \frac{k_i}{1++rk_i}, 1 \leq i \leq n.$

## Proof:

We prove the theorem by induction method.

**a)** If $X_p$ is spacelike $\varepsilon_1$=+1. For $n = 1$, the theorem holds. Actually,

$$\varphi(r, \overline{k}_1, \overline{k}_2, \dots, \overline{k}_n) = \prod_{i=1}^{1} (1 + \varepsilon_i r \overline{k_\iota}) = 1 + r\overline{k}_1 = 1 + r \sum_{i=1}^{1} \varepsilon_i \overline{k_\iota} = 1 + r\overline{K_1}$$

Now suppose that the theorem holds for $n-1$ and show that is true for $n$:

$$\varphi(\mathrm{r}, \bar{\mathrm{k}}_1, \bar{\mathrm{k}}_2, \ldots, \bar{k}_n) = \prod_{i=1}^{n-1}(1+\varepsilon_i r\bar{k}_i)$$
$$= 1 + r\sum_{i=1}^{n-1}\bar{k}_i + r^2\sum_{i<j}^{n-1}\bar{k}_i\,\bar{k}_j + \cdots + r^{n-1}\prod_{i=1}^{n-1}\bar{k}_i$$
$$= 1 + \mathrm{r}\overline{K_1} + r^2\overline{K_2} + \cdots + r^{n-1}\overline{K}_{n-1}$$

For $n$, both sides of the equation is multiplied by $1 + r\overline{k_n}$

$$\left(\prod_{i=1}^{n-1} 1 + r\bar{k}_i\right)(1 + r\overline{k_n}) = \left(1 + r\sum_{i=1}^{n-1}\bar{k}_i + r^2\sum_{i<j}^{n-1}\bar{k}_i\,\bar{k}_j + \cdots + r^{n-1}\prod_{i=1}^{n-1}\bar{k}_i\right)(1 + r\overline{k_n})$$
$$= 1 + r\left(\sum_{i=1}^{n-1}\bar{k}_i + \overline{k_n}\right) + r^2\left(\sum_{i<j}^{n-1}\bar{k}_i\,\bar{k}_j + \overline{k_n}\sum_{i=1}^{n-1}\bar{k}_i\right) + \cdots + r^n\overline{k_n}\prod_{i=1}^{n-1}\bar{k}_i$$

and we have

$$\prod_{i=1}^{n} 1 + r\bar{k}_i = 1 + r\sum_{i=1}^{n}\bar{k}_i + r^2\sum_{i<j}^{n}\bar{k}_i\,\bar{k}_j + \cdots + r^n\prod_{i=1}^{n}\bar{k}_i$$
$$\varphi(\mathrm{r}, \bar{\mathrm{k}}_1, \bar{\mathrm{k}}_2, \ldots, \bar{k}_n) = 1 + \mathrm{r}\overline{K_1} + r^2\overline{K_2} + \cdots + r^n\overline{K}_n$$

**b)** If $X_p$ is timelike $\varepsilon_1$=-1. For $n = 1$, the theorem holds . Actually,

$$\varphi(\mathrm{r}, \bar{\mathrm{k}}_1, \bar{\mathrm{k}}_2, \ldots, \bar{k}_n) = \prod_{i=1}^{1}(1+\varepsilon_i r\bar{k}_i)$$
$$= 1 + \varepsilon_1 r\bar{\mathrm{k}}_1$$
$$= 1 + r\sum_{\mathrm{i}=1}^{1}\varepsilon_{\mathrm{i}}\bar{\mathrm{k}}_{\mathrm{i}}$$
$$= 1 + r\overline{K_1}$$

Now suppose that the theorem holds for $n-1$ and show that is true for $n$:

$$\varphi(\mathrm{r}, \bar{\mathrm{k}}_1, \bar{\mathrm{k}}_2, \ldots, \bar{k}_n) = \prod_{i=1}^{n-1} 1+\varepsilon_i r\bar{k}_i$$
$$= 1 + r\sum_{i=1}^{n-1}\varepsilon_i\bar{k}_i + r^2\sum_{i<j}^{n-1}\varepsilon_i\,\bar{k}_i\,\bar{k}_j + \cdots + r^{n-1}\prod_{i=1}^{n-1}\varepsilon_i\,\bar{k}_i$$
$$= 1 + \mathrm{r}\overline{K_1} + r^2\overline{K_2} + \cdots + r^{n-1}\overline{K}_{n-1}$$

For $n$, both sides of the equation is multiplied by $1 + r\varepsilon_n\overline{k_n}$

$$\left(\prod_{i=1}^{n-1} 1 + r\varepsilon_i\overline{k_\iota}\right)\left(1 + r\varepsilon_i\overline{k_n}\right)$$

$$= \left(1 + r\sum_{i=1}^{n-1}\varepsilon_i\overline{k_\iota} + r^2\sum_{i<j}^{n-1}\varepsilon_i\overline{k_\iota}\,\overline{k_j} + \cdots + r^{n-1}\prod_{i=1}^{n-1}\varepsilon_i\overline{k_\iota}\right)\left(1 + r\varepsilon_n\overline{k_n}\right)$$

$$= 1 + r\left(\sum_{i=1}^{n-1}\varepsilon_i\overline{k_\iota} + \overline{\varepsilon_n k_n}\right) + r^2\left(\sum_{i<j}^{n-1}\varepsilon_i\overline{k_\iota}\,\overline{k_j} + \overline{k_n}\sum_{i=1}^{n-1}\varepsilon_i\overline{k_\iota}\right) + \cdots + r^n\varepsilon_n\,\overline{k_n}\prod_{i=1}^{n-1}\varepsilon_i\overline{k_\iota}$$

and we have

$$\varphi\left(\mathrm{r}, \bar{\mathrm{k}}_1, \bar{\mathrm{k}}_2, \ldots, \bar{k}_n\right) = 1 + \mathrm{r}\overline{K_1} + r^2\overline{K_2} + \cdots + r^n\overline{K}_n$$

## 4. Conclusion

Consequently, in this paper we studied higher order Gaussian curvatures on $\bar{M}$ in $E_1^{n+1}$ and we proved a new theorem that related higher order Gaussian curvatures and principal curvatures.

## Competing Interests

Authors have declared that no competing interests exist.

## References


[1] O.Neill , B., Semi Riemannian Geometry. Department of Mathematics Uni-versity of California Los Angeles, California. 1983

[2] Sağel M.K.and Hacısalihoğlu, H.H.1988. On the Gaussian and mean curva-tures of a paralel hypersurface I: Commun. Fac. Sci.Univ. Ankara, Ser. Al 37,No.1-2,9-1500.

[3] Yaşar, A. Higher Order Gaussian Curvatures of a Parallel Hypersurfaces in Ln Lorentz Space, Master Thesis Ankara University.





***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*www.sciencedomain.org/review-history.php?iid=360&id=6&aid=2593*