Fast Fourier Transform of Multi-Assets Options under Economic Recession Induced Uncertainties

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Abstract

A Fast Fourier transform approach has been presented by Carr & Madan (2009) on a single underlying asset. In this current research paper, we present fast Fourier transform algorithm for the valuation of Multi-asset Options under Economic Recession Induced Uncertainties. The issue of multi-dimension in both finite and infinite case of Options is part of the focus of this research. The notion of economic recession was incorporated. An intuition behind the introduction of recession induced volatility uncertainty is revealed by huge volatility variation during the period of economic recession compared to the period of recession-free. Nigeria economic recession outbreak in 2016 and its effects on the uncertainty of the payoffs of Nigeria Stocks Exchange (NSE) among other investments was among the motivating factors for proposing economic recession induced volatility in options pricing. The application of the proposed Fast Fourier Transform algorithm in handling multi-assets options was shown. A new result on options pricing was achieved and capable of yielding efficient option prices during and out of recession. Numerical results were presented on assets in 3-dimensions as an illustration taking Black Scholes prices as a bench mark for method effectiveness comparison. The key findings of this research paper among other crucial contributions could be seen in computational procedure of options valuation in multi-dimensions and uncertainties in options payoffs under the exposure of economic recession.

Keywords

Fast Fourier Transform (FFT), Multi-Assets, Finite and Infinite Dimension of Assets, Economic Recession, Volatility Change, European Options

1. Introduction

In the history of options valuation, the work of Black-Scholes [1] (1973) cannot

be forgotten no matter the shortcomings we may find in their work such as the issue of constant values assumed by the volatility and interest rates. Divers financial assets are traded in financial market. Some are risky assets while some are riskfree. Multi-asset options generally are a set of options whose pay-offs are governed by more than one underlying assets. A typical example of multi-asset option is a basket option. Basket option is based on two or more underlying assets or stocks. There is European type basket call or put option whereby the overall value of a basket of assets plays the role of the price of the underlying asset in single-asset European. Likewise, we have American style basket options as well. Other types of multi-assets options include: spread option, exchange options, Quanto options, Rainbow options, etc. Various types of multi-asset options could be described based on their payoff function. Beom, J. K. *et al.* (2013) [2] presents a simple numerical approach for American put options valuation.

The focus of most researchers is limited to two dimensions in the valuation of multi-asset options. Some authors such as Yuwei Chen [3] gave their formulation in terms of Black Scholes Partial Differential Equation (PDE) and considered numerical method for obtaining the prices. The authors, Carol Alexander *et al.*, (2009) [4] did not relent in their effort as they presented an analytic approximation for Multi-asset Options Pricing but their approach was different from the one we gave here. In recent time, an existence result of solutions for a multi-dimensional stochastic control problem with singular state constraint was presented by Ulrich Horst and Xiaonyu Xia, (2018) [5]. The problem at hand is not centered on the specialty of the type of multi-asset but how to handle the nature of multi-assets amidst other conditional forces of uncertainty and dimensional challenges in the valuation process. According to Yuwei Chen [3], most multi-asset European options and all American options have no analytical formula giving their prices, therefore, numerical methods are employed to approximate such prices.

We consider challenges of uncertainty posed by instability of economic factors on the payoff of assets. The keen economic factor to us is recession. We consider volatility variation imposed on options prices during economic recession which we termed economic recession induced volatility in addition to other volatility uncertainties of options price from other sources. Fast Fourier Transform algorithm is used in our formulation. Many authors have worked on Fourier transform of some options valuation such as ([6] [7] [8] [9]) among others. We gave a framework for extending Carr & Madan [6] Fast Fourier Transform approach of valuating an option on a single underlying asset to valuation of multi-assets options. Our major contribution is seen in valuation of multi-assets option in both finite and infinite dimensions and incorporation of economic recession factor on the volatility of assets during the state of economic recession.

The rest part of this paper is organized as follows. In Section 2, we gave preliminaries on Fast Fourier Transform. Section 3 is the Main result, which consists of subsections Fast Fourier Transform of multi-assets in finite dimension

and Fast Fourier Transform of multi-assets in infinite dimension. Section 4 is the application on three assets in 3-dimensional case while Section 5 shows Numerical results on multi-assets in 3-dimensions. The last section is the conclusion part.

2. Preliminaries

2.1. Some Definitions on Fourier Transform

Definition 1: *Let* $f : R \to R$ be a real-value function for $x \in R$, then the Fourier transform of *f* is defined as

$$\mathcal{F}\left(f\left(x\right);w\right) = \hat{f}\left(w\right) = \int_{-\infty}^{\infty} f\left(x\right) e^{-i\omega x} \mathrm{d}x, \ \omega \in R, \tag{1}$$

where $i = \sqrt{-1}$ and ω is a parameter. We can recover $f\left(x\right)$ from $f\left(\omega\right)$ by the inverse Fourier transform.

Definition 2: *The inverse Fourier transform of the real valued function* $f\left(x\right)$ *is defined by*

$$\mathcal{F}^{-1}\left(\hat{f}\left(\omega\right);x\right) = f\left(x\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}\left(w\right) e^{i\omega x} \mathrm{d}\omega, \ x \in R \tag{2}$$

which belongs to L^2-spaces.

Some certain axioms on Fourier transform exist in literature. Among such axioms are: convolution axiom, Translation axiom, Linearity axiom, Differentiation axiom, Symmetric axiom etc.

2.2. Some Axioms of Fourier Transform

- **Convolution axiom:** This property is considered very crucial as convolution of functions is mapped into multiplication in Fourier space.

  Let $f, g \in L^2$-space.

  Then their convolution is defined by

$$\left(f \star g\right)\left(x\right) = \int_{-\infty}^{\infty} f\left(x-y\right) g\left(y\right) \mathrm{d}y = \int_{-\infty}^{\infty} f\left(y\right) g\left(x-y\right) \mathrm{d}y \tag{3}$$

The Fourier transform of the convolution of the functions *f* and *g* with convolution operator $\star$ is given by

$$\mathcal{F}\left(\left(f \star g\right);\omega\right) = \int_{-\infty}^{\infty} \left[f\left(x-y\right) g\left(y\right) \mathrm{d}y\right] e^{-i\omega x} \mathrm{d}x = \hat{f}\left(\omega\right) \hat{g}\left(\omega\right) \tag{4}$$

The proof of Equation (4) is given in what follows.

Proof:

$$\mathcal{F}\left(\left(f \star g\right);\omega\right) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(x-y\right) g\left(y\right) \mathrm{d}y\right] e^{-i\omega x} \mathrm{d}x \tag{5}$$

$$\mathcal{F}\left(\left(f \star g\right);\omega\right) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(x-y\right) g\left(y\right) e^{-i\omega x} \mathrm{d}y\right] \mathrm{d}x \tag{6}$$

$$\mathcal{F}\left(\left(f \star g\right);\omega\right) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(x-y\right) g\left(y\right) e^{-i\omega x} \mathrm{d}x\right] \mathrm{d}y \tag{7}$$

$$\mathcal{F}\big((f \star g); \omega\big) = \int_{-\infty}^{\infty} g(y) \left[\int_{-\infty}^{\infty} f(x-y) e^{-i\omega x} dx \right] dy \tag{8}$$

To this end, one will have

$$\mathcal{F}\big((f \star g); \omega\big) = \int_{-\infty}^{\infty} g(y) e^{-i\omega y} \left[\int_{-\infty}^{\infty} f(\zeta) e^{-i\omega(\zeta+y)} d\zeta \right] dy \tag{9}$$

$$\mathcal{F}\big((f \star g); \omega\big) = \int_{-\infty}^{\infty} g(y) e^{-i\omega(y)} \left[\int_{-\infty}^{\infty} f(\zeta) e^{-i\omega \zeta} d\zeta \right] dy \tag{10}$$

$$\mathcal{F}\big((f \star g); \omega\big) = \hat{f}(\omega) \int_{-\infty}^{\infty} g(y) e^{-i\omega y} dy \tag{11}$$

$$\mathcal{F}\big((f \star g); \omega\big) = \hat{f}(\omega) \hat{g}(\omega). \tag{12}$$

Similarly,

$$\mathcal{F}(fg; \omega) = \mathcal{F}\big(f(x;\omega)\big) \star \mathcal{F}\big(g(x;\omega)\big) \tag{13}$$

$$\mathcal{F}(fg; \omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \star \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \tag{14}$$

$$\mathcal{F}(fg; \omega) = \hat{f}(\omega) \star \hat{g}(\omega). \tag{15}$$

The notion of Fourier transform is related to characteristic function in the following sense.

Definition 3: *Suppose $P(x)$ is a probability distribution function with $x \in R$. Then a characteristic function $\phi(\omega), \omega \in IR$ is defined as the Fourier transform of $P(x)$ by*

$$\mathcal{F}\big(P(x); \omega\big) \equiv \hat{\phi}(\omega) \equiv \int_{-\infty}^{\infty} P(x) e^{i\omega x} d\omega = \mathbb{E}\big[e^{i\omega x}\big] \tag{16}$$

One can recover the probability distribution function $P(x)$ by taking the inverse Fourier transform of the characteristics function just as the definition of inverse Fourier transform presented in Equation (2).

This is given by

$$P(x) = \mathcal{F}^{-1}\big(\hat{\phi}(\omega)\big) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\omega) e^{-i\omega x} d\omega \tag{17}$$

The reader can see the reference ([10]) and other standard texts on Fourier transform for more information.

There are available models for predicting economic recession probabilities especially on US recession (See [11]). The application of probability theory is found useful in such estimation [12]. Philip *et al.* [13] gave an intuition in predicting price dynamics in the presence of economic recession via recurrence relation technique and further presented a simple model showing the future of an economy under the exposure of recession if strong strategies and policies were not put in place to revive the economy. Fabio, Fornari and Antonio, Mele in (2009) [14] showed some connections between Financial Volatility and Economic Activity.

In sequel, we examine uncertainty on options value pose by recession on the volatility parameter of an option.

Assumption: Supposed that there are two sources of volatility variation on the assets. These are classified into economic recession induced volatility, v^{rec},

and the usual volatility from other sources, v^*, all assumed to be stochastic in nature. The volatility of the options is then given by

$$V(t) = \begin{cases} v^{rec} + v^*, & \text{if the economy is in recession;} \\ v^*, & \text{if economy is recession free.} \end{cases} \tag{18}$$

where v^{rec} is economy recession induced volatility in the stock market which arises due to information flow from the recessed Economy while v^* is the volatility in the market from other sources. The study of parameters uncertainties in some financial models has received attention in the past few years. Two of those authors are [15] and [16]. Unfortunately, many authors are yet to consider uncertaianties parameters that focus on economic recession in most Financial models use especially in derivative pricing. This present research is believed to stand out in that regards as it is hoped that more attention of authors will be drawn to economic recession and its effects in options pricing.

3. The Main Result

1) The Fast Fourier Transform (FFT) algorithm goes thus.

2) First get the characteristic function of the distribution. 2. Derive an analytic expression for the Fourier Transform of the modified options price wrt the characteristic function.

3) Find the Fourier inversion of the pricing function.

4) Perform Discrete Fourier transform in both step 2 and 3 above and apply Trapezoidal rule.

5) Choose uniform grid size for the computation of the FFT in step 4.

6) Make choices on the values assumed by other parameters such as N, the decay rate (α), etc. for optimality purpose.

3.1. Fast Fourier Transform of Multi-Assets in Finite Dimension

Consider an n-factor underlying asset $S_j, j = 1, 2, \cdots, n$. The payoff function $H(S, K)$ of the options at maturity time on the given underlying multi-assets is defined as

$$H(S, K) = \prod_{j=1}^{n} \left(c(S_j - K_j)^+ \right). \tag{19}$$

$$H(S, K) = \left(c(S_1 - K_1)^+ \cdot c(S_2 - K_2)^+ \cdot c(S_3 - K_3)^+ \cdots c(S_n - K_n)^+ \right) \tag{20}$$

where the S_j and K_j are respectively asset and strike prices.
The

$$(S_j - K_j)^+ = \left(\max(S_j - K_j, 0), \text{ for each } j = 1, 2, \cdots, n \right)$$

The parameter c is a control switch between *call or put* options. If c as defined in the Equations (19) and (20) is -1, then one has put option and if $c = +1$, one has call option.

Let $k_j, s_j \mid j = 1, 2, \cdots, n$ be logarithm of the strikes K_j and assets S_j prices

respectively. Then, the option values computation in n-dimension is given by

$$
\begin{aligned}
&V_T\left(k_1, k_2, \cdots, k_n\right) \\
&= E^{\mathcal{Q}}\left(\mathrm{e}^{-rT}\left(S_1(T) - K_1(T)\right)^+ \cdot \left(S_2(T) - K_2(T)\right)^+ \cdots \left(S_n(T) - K_n(T)\right)^+\right)
\end{aligned}
\tag{21}
$$

where $\mathcal{Q}$ is taken to be risk neutral measure while q_T is the joint density function of each $s_j(T), j = 1, 2, \cdots, n$.

The integral representation of Equation (21) is given as

$$
\begin{aligned}
&V_T\left(k_1, k_2, \cdots, k_n\right) \\
&= \int_{k_1}^{\infty} \int_{k_2}^{\infty} \cdots \int_{k_n}^{\infty} \mathrm{e}^{-rT}\left(\mathrm{e}^{s_1} - \mathrm{e}^{k_1}\right)\left(\mathrm{e}^{s_2} - \mathrm{e}^{k_2}\right) \cdots \left(\mathrm{e}^{s_n} - \mathrm{e}^{k_n}\right) \\
&\quad \times q_T\left(s_1, s_2, \cdots, s_n\right) \mathrm{d}s_n \cdots \mathrm{d}s_2 \mathrm{d}s_1
\end{aligned}
\tag{22}
$$

We gave the characteristic function corresponding to the joint density as

$$
\varphi\left(u_1, u_2, \cdots, u_n\right) = E^{\mathcal{Q}}\left(\mathrm{e}^{\sum iu_j s_j(T)}\right)
\tag{23}
$$

$$
\varphi\left(u_1, u_2, \cdots, u_n\right) = \int_{-\infty}^{\infty} \mathrm{e}^{iu_1 s_1 + iu_2 s_2 + \cdots + iu_n s_n} q_T\left(s_1, s_2, \cdots, s_n\right) \mathrm{d}s_n \cdots \mathrm{d}s_2 \mathrm{d}s_1.
\tag{24}
$$

In order to ensure the square integrability of the right hand side of the Equation (22), we take its product and a decaying term α over k_j.

This we expressed as

$$
c_T\left(k_1 k_2, \cdots, k_n\right) = \mathrm{e}^{\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_n k_n} \times V_T\left(k_1, k_2, \cdots, k_n\right), \quad \alpha_1, \alpha_2, \cdots, \alpha_n > 0.
\tag{25}
$$

For the call option, the Fourier transform of the modified option prices is given by

$$
\begin{aligned}
&\psi_T\left(u_1, u_2, \cdots, u_n\right) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{e}^{i\left(u_1 k_1 + u_2 k_2 + \cdots + u_n k_n\right)} c_T\left(k_1, k_2, \cdots, k_n\right) \mathrm{d}k_n \cdots \mathrm{d}k_2 \mathrm{d}k_1
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\psi_T\left(u_1, u_2, \cdots, u_n\right) \\
&= \int \cdots \int_{R^n} \mathrm{e}^{\left(\alpha_1 + iu_1\right)k_1 + \left(\alpha_2 + iu_2\right)k_2 + \cdots + \left(\alpha_n + iu_n\right)k_n} \\
&\quad \times \int_{k_n}^{+\infty} \cdots \int_{k_2}^{+\infty} \int_{k_1}^{+\infty} \mathrm{e}^{-rT}\left(\mathrm{e}^{s_1} - \mathrm{e}^{k_1}\right)\left(\mathrm{e}^{s_2} - \mathrm{e}^{k_2}\right) \cdots \left(\mathrm{e}^{s_n} - \mathrm{e}^{k_n}\right) \\
&\quad \times q_T\left(s_1, s_2, \cdots, s_n\right) \mathrm{d}k_n \cdots \mathrm{d}k_2 \mathrm{d}k_1 \mathrm{d}s_n \cdots \mathrm{d}s_2 \mathrm{d}s_1
\end{aligned}
\tag{27}
$$

This is further simplified as follows

$$
\begin{aligned}
&\psi_T\left(u_1, u_2, \cdots, u_n\right) \\
&= \int \cdots \int_{R^n} \mathrm{e}^{-rT} q_T\left(s_1, s_2, \cdots, s_n\right) \\
&\quad \times \int_{\infty}^{s^n} \cdots \int_{\infty}^{s^2} \int_{\infty}^{s^1} \mathrm{e}^{-rT}\left(\left(\mathrm{e}^{s_1} - \mathrm{e}^{k_1}\right)\left(\mathrm{e}^{s_2} - \mathrm{e}^{k_2}\right) \cdots \left(\mathrm{e}^{s_n} - \mathrm{e}^{k_n}\right)\right) \\
&\quad \times \mathrm{e}^{\left(\alpha_1 + iu_1\right)k_1 + \left(\alpha_2 + iu_2\right)k_2 + \cdots + \left(\alpha_n + iu_n\right)k_n} \mathrm{d}k_n \cdots \mathrm{d}k_2 \mathrm{d}k_1 \mathrm{d}s_n \cdots \mathrm{d}s_2 \mathrm{d}s_1
\end{aligned}
\tag{28}
$$

which reduces to

$$
\begin{aligned}
&\psi_T\left(u_1, u_2, \cdots, u_n\right) \\
&= \int \cdots \int_{R^n} \mathrm{e}^{-rT} q_T\left(s_1, s_2, \cdots, s_n\right) \\
&\quad \times \left(\frac{\left(\mathrm{e}^{\left(1+\alpha_1+iu_1\right)s_1 + \left(1+\alpha_2+iu_2\right)s_2 + \cdots + \left(1+\alpha_n+iu_n\right)s_n}\right)}{\left(\alpha_1 + iu_1\right)\left(1 + \alpha_1 + iu_1\right)\left(\alpha_2 + iu_2\right)\left(1 + \alpha_2 + iu_2\right) \cdots \left(\alpha_n + iu_n\right)\left(1 + \alpha_n + iu_n\right)}\right) \mathrm{d}s_n \cdots \mathrm{d}s_2 \mathrm{d}s_1
\end{aligned}
\tag{29}
$$

For a known characteristic function or a calculated affine characteristic function φ_T of a given process, it is possible to derive an analytic close form formula for $\Psi_T(u_1, u_2, \cdots, u_n)$ which is the Fourier transform of the multi-assets.

Applying definition (2), we expressed the Fourier transform of (21) through (28)-(29) as

$$
\begin{aligned}
&V_T(k_1, k_2, \cdots, k_n) \\
&= \frac{e^{-(\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_n k_n)}}{(2\pi)^n} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n\text{-tuple}} e^{-i(u_1 k_1 + u_2 k_2 + \cdots + u_n k_n)} \psi_T(u_1, u_2, \cdots, u_n) \, du_k \cdots du_2 du_1.
\end{aligned} \tag{30}
$$

Rewriting the integrals in series form using trapezoidal rule, equation (30) is transformed to

$$
\begin{aligned}
&V_T(k_1, k_2, \cdots, k_n) \\
&= \frac{e^{-(\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_n k_n)}}{(2\pi)^n} \underbrace{\sum_{m_1=1}^{N_1} \sum_{m_2=1}^{N_2} \cdots \sum_{m_n=1}^{N_n}}_{n\text{-tuple}} e^{-i(u_{1,m_1} k_1 + u_{2,m_2} k_2)} \psi_T(u_{1,m_1}, u_{2,m_2}, \cdots, u_{n,m_n}) \prod_{j=1}^{n} h_j
\end{aligned} \tag{31}
$$

$$\underbrace{\hspace{6cm}}_{\Omega(k_1, k_2, \cdots, k_n)}$$

where

$\prod_{j=1}^{n} h_j = h_1 h_2 \cdots h_n$, while $h_j, j = 1, 2, \cdots, n$ denote integration steps here

For a single asset, Fast Fourier Transform (FFT) equation is defined in the form of

$$
H(k) = \sum_{j=1}^{N-1} e^{-i \frac{2\pi}{N}(j-1)(k-1)} y_j, \quad 1 \le k \le N. \tag{32}
$$

Therefore, in the light of Equation (32), we gave an n-dimension version of $y_j, j = 1, 2, \cdots, n$, such that in an effort to capture the imaginary part of the complex form of y_j, which is inclusive in Fourier transform, we have

$$
\xi_j \mid j = 1, 2, \cdots, n := \sum_{m_1=1}^{N_1-1} \sum_{m_2=1}^{N_1-1} \cdots \sum_{m_n=1}^{N_1-1} e^{-i\left(\frac{2\pi}{N_1} j_1 m_1 + \frac{2\pi}{N_2} j_2 m_2 + \cdots + \frac{2\pi}{N_n} j_n m_n\right)} y_j, \quad j = 1, 2, \cdots, n \tag{33}
$$

where $m_1 = 1, 2, \cdots, N_1 - 1; m_2 = 1, 2, \cdots, N_2 - 1$ respectively.

The integration steps $h_j, j = 1, 2, \cdots, n$, in (31) is uniformly chosen such that

$$
h_1 = \frac{u_{1,m_1}}{\left(m_1 - \frac{N}{2}\right)}, h_2 = \frac{u_{2,m_2}}{\left(m_2 - \frac{N}{2}\right)}, \cdots, h_n = \frac{u_{n,m_n}}{\left(m_n - \frac{N}{2}\right)}
$$

Considering a uniform grid size of $N \times N$ array and set.

The size of the uniform spacing ζ between the N values of vector k with respect to the integration steps $h_j, j = 1, 2, \cdots, n$ and Equation (33) has the following relation

$$
\zeta_1 h_1 = \zeta_2 h_2 = \cdots = \zeta_n h_n = \frac{2\pi}{N} \tag{34}
$$

The options prices V_T over log strikes from Equation (31) is now given as

$$V_T\left(k_{1,p_1}, k_{2,p_2}, \cdots, k_{n,p_n}\right)$$

$$\approx \frac{e^{-\left(\alpha_1 k_{1,p_1} + \alpha_2 k_{2,p_2} + \cdots + \alpha_n k_{n,p_n}\right)}}{(2\pi)^n} \Omega\left(k_{1,p_1}, k_{2,p_2}, \cdots, k_{n,p_n}\right) \prod_{j=1}^{n} h_j, \tag{35}$$

$$\text{for } 0 \le p_1, p_2, \cdots, p_n \le N-1.$$

where

$$\Omega\left(k_{1,p_1}, k_{2,p_2}, \cdots, k_{n,p_n}\right)$$

$$= \sum_{m_1=1}^{N_1-1} \sum_{m_2=1}^{N_1-1} \cdots \sum_{m_n=1}^{N_1-1} e^{-\frac{2\pi}{N}\left(\left(m_1-\frac{N}{2}\right)\left(p_1-\frac{N}{2}\right) + \left(m_2-\frac{N}{2}\right)\left(p_2-\frac{N}{2}\right) + \cdots + \left(m_n-\frac{N}{2}\right)\left(p_n-\frac{N}{2}\right)\right)} \psi_T\left(u_1, u_2, \cdots, u_n\right)$$

Remarks:

1) The Equation (35) above is fast Fourier transform formula which can be adopted to calculate the value of European call option prices on finite n-dimensional underlying multi-assets.

2) The corresponding European put option value can be obtained if we set the control parameter $c = -1$ in the Equations (19)-(20) such that the sign attached to the log strikes and log assets changes in Equation (21) follow the same procedure through to arrive at (35). Alternatively, the put values can be obtained via put-call parity.

3.2. Fast Fourier Transform of Multi-Assets in an Infinite Dimension

Following similar algorithm presented in finite dimensional case, the above formulation can be extended to an infinite dimension as follows.

The payoff function $H(S,K)$ of the options at maturity time on the given underlying multi-assets in an *infinite dimension* is given as

$$H(S,K) = \prod_{j=1}^{\infty}\left(c\left(S_j - K_j\right)\right) \tag{36}$$

$$H(S,K)$$
$$= \left(c\left(S_1 - K_1\right)^+ \cdot c\left(S_2 - K_2\right)^+ \cdot c\left(S_3 - K_3\right)^+ \cdots c\left(S_n - K_n\right)^+ \cdot c\left(S_{n+1} - K_{n+1}\right)^+ \cdots\right) \tag{37}$$

where the S_j and K_j are respectively asset and strike prices. The parameter c is a control in switching between *call or put* options as discussed earlier.

Let $k_j, s_j \mid j = 1, 2, \cdots, n, n+1, \cdots$ be logarithm of the strikes K_j and assets S_j prices respectively. Then, the option values computation in infinite-dimension is given by

$$V_T\left(k_1, k_2, \cdots, k_{n+1}, \cdots\right)$$
$$= E^{\mathcal{Q}}\left(e^{-rT}\left(S_{1_T} - K_{1_T}\right)^+ \cdot \left(S_{2_T} - K_{2_T}\right)^+ \cdots \left(S_{n_T} - K_{n_T}\right)^+ \cdot \left(S_{n+1_T} - K_{n+1_T}\right)^+ \cdots\right) \tag{38}$$

where $\mathcal{Q}$ still remain as risk neutral measure and q_T is the joint density function for each

$$s_j(T), j = 1, 2, \cdots, n, n+1, \cdots$$

The characteristic function corresponding to the joint density is given as

$$\varphi\left(u_1, u_2, \cdots, u_{n+1}, \cdots\right) = E^{Q}\left(e^{\sum_j i u_j s_j(T)}\right)$$

$$\varphi\left(u_1, u_2, \cdots, u_{n+1}, \cdots\right)$$
$$= \int_{-\infty}^{\infty} e^{i u_1 s_1 + i u_2 s_2 + \cdots + i u_{n+1} s_{n+1} + \cdots} q_T\left(s_1, s_2, \cdots, s_{n+1}, \cdots\right) ds_{n+1} \cdots ds_2 ds_1.$$

The Fourier transform of the modified call option prices is given by

$$\varphi\left(u_1, u_2, \cdots, u_{n+1}, \cdots\right)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\left(u_1 k_1 + u_2 k_2 + \cdots + u_{n+1} k_{n+1} + \cdots\right)} c_T\left(k_1, k_2, \cdots, k_{n+1}, \cdots\right) dk_{n+1} \cdots dk_2 dk_1 \tag{39}$$

$$\psi_T\left(u_1, u_2, \cdots, u_{n+1}, \cdots\right)$$
$$= \int \cdots \int_{IR^n} e^{\left(\alpha_1 + i u_1\right) k_1 + \left(\alpha_2 + i u_2\right) k_2 + \cdots + \left(\alpha_{n+1} + i u_{n+1}\right) k_{n+1} + \cdots}$$
$$\times \int_{k_{n+1}}^{+\infty} \cdots \int_{k_2}^{+\infty} \int_{k_1}^{+\infty} e^{-rT}\left(\left(e^{s_1} - e^{k_1}\right)\left(e^{s_2} - e^{k_2}\right) \cdots \left(e^{s_{n+1}} - e^{k_{n+1}}\right) \cdots\right) \tag{40}$$
$$\times q_T\left(s_1, s_2, \cdots, s_{n+1}, \cdots\right) dk_{n+1} \cdots dk_2 dk_1 ds_{n+1} \cdots ds_2 ds_1$$

Further simplification yields

$$\psi_T\left(u_1, u_2, \cdots, u_{n+1}, \cdots\right)$$
$$= \int \cdots \int e^{-rT} q_T\left(s_1, s_2, \cdots, s_{n+1}, \cdots\right)$$
$$\times \int_{\infty}^{s_2} \int_{\infty}^{s_1} e^{-rT}\left(\left(e^{s_1} - e^{k_1}\right)\left(e^{s_2} - e^{k_2}\right) \cdots \left(e^{s_{n+1}} - e^{k_{n+1}}\right) \cdots\right) \tag{41}$$
$$\times e^{\left(\alpha_1 + i u_1\right) k_1 + \left(\alpha_2 + i u_2\right) k_2 + \cdots + \left(\alpha_{n+1} + i u_{n+1}\right) k_{n+1} + \cdots} dk_{n+1} \cdots dk_2 dk_1 ds_{n+1} \cdots ds_2 ds_1$$

which reduces to

$$\psi_T\left(u_1, u_2, \cdots, u_{n+1}, \cdots\right)$$
$$= \int \cdots \int_{R^\infty} e^{-rT} q_T\left(s_1, s_2, \cdots, s_{n+1}, \cdots\right) \tag{42}$$
$$\times \left(\frac{\left(e^{\left(1 + \alpha_1 + i u_1\right) s_1 + \left(1 + \alpha_2 + i u_2\right) s_2 + \cdots + \left(1 + \alpha_{n+1} + i u_{n+1}\right) s_{n+1} + \cdots}\right)}{\left(\alpha_1 + i u_1\right)\left(1 + \alpha_1 + i u_1\right)\left(\alpha_2 + i u_2\right)\left(1 + \alpha_2 + i u_2\right) \cdots \left(\alpha_{n+1} + i u_{n+1}\right)\left(1 + \alpha_{n+1} + i u_{n+1}\right) \cdots}\right) \cdots ds_{n+1} \cdots ds_2 ds_1$$

Expressing the integrals in discrete Fourier transform and using trapezoidal rule, we have

$$V_T\left(k_1, k_2, \cdots, k_{n+1}, \cdots\right) = \frac{e^{-\left(\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_{n+1} k_{n+1} + \cdots\right)}}{\left(2\pi\right)^{n+\nu}}$$
$$\times \sum_{m_1=1}^{N_1} \sum_{m_2=1}^{N_2} \cdots \sum_{m_\infty}^{N_\infty} e^{-i\left(u_{1,m_1} k_1 + u_{2,m_2} k_2 + \cdots\right)} \psi_T\left(u_{1,m_1}, u_{2,m_2}, \cdots, u_{n+1,m_{n+1}} \cdots\right) \prod_{j=1}^{\infty} h_j \tag{43}$$

where
$$\prod_{j=1}^{\infty} h_j = h_1 h_2 \cdots h_{n+1} \cdots, \text{ while } h_j, \quad j = 1, 2, \cdots, n+1, \cdots \text{ denote integration steps}$$
here and ν is a very large integer.

The options prices V_T over log strikes $k = \log K$ are finally given as

$$V_T\left(k_{1,p_1}, k_{2,p_2} \cdots, k_{n+1,p_{n+1}}, \cdots\right)$$
$$\approx \frac{e^{-\left(\alpha_1 k_{1,p_1} + \alpha_2 k_{2,p_2} + \cdots + \alpha_{n+1} k_{n+1,p_{n+1}} + \cdots\right)}}{\left(2\pi\right)^{n+\nu}} \Omega\left(k_{1,p_1}, k_{2,p_2}, \cdots, k_{n,p_n}\right) \prod_{j=1}^{\infty} h_j \tag{44}$$

where $0 \le p_1, p_2, \cdots, p_{n+1}, \cdots \le N-1$ and

$$\Omega\left(k_{1,p_1}, k_{2,p_2}, \cdots, k_{n+1,p_{n+1}}, \cdots\right)$$
$$= \sum_{m_1=1}^{N_1-1} \sum_{m_2=1}^{N_1-1} \cdots \sum_{m_{n+1}=1}^{N_1-1} \cdots e^{-\frac{2\pi}{N}\left(\left(m_1-\frac{N}{2}\right)\left(p_1-\frac{N}{2}\right) + \left(m_2-\frac{N}{2}\right)\left(p_2-\frac{N}{2}\right) + \cdots + \left(m_{n+1}-\frac{N}{2}\right)\left(p_{n+1}-\frac{N}{2}\right)\right)}$$
$$\times \psi_T\left(u_1, u_2, \cdots, u_{n+1} \cdots\right)$$

We acknowledge Carr & Madan (1999) [6] for providing a fast Fourier Transform framework on single underlying asset which we extended to multi-assets in both *finite dimension and infinite dimension* in this paper.

4. Application on Three Assets in 3-Dimensional Case

Let $S_j \mid j = 1, 2, 3$ be three underlying assets and strike prices $K_j \mid j = 1, 2, 3$ respectively. The payoff function is defined in the light of Equation (20) as

$$H(S,K) = \left(c(S_1 - K_1)^+ \cdot c(S_2 - K_2)^+ \cdot c(S_3 - K_3)^+\right), \quad c = +1 \text{ for call option.} \quad (45)$$

where the S_j and K_j are respectively asset and strike prices.

The option values computation in 3-dimension is given by

$$V_T(k_1, k_2, k_3)$$
$$= E^Q\left(e^{-rT}\left(S_1(T) - K_1(T)\right)^+ \cdot \left(S_2(T) - K_2(T)\right)^+ \cdot \left(S_3(T) - K_3(T)\right)^+\right) \quad (46)$$

where Q is a risk neutral measure and q_T is the joint density function of each $s_j(T), j = 1, 2, 3$.

The integral representation of Equation (46) is given as

$$V_T(k_1, k_2, k_3)$$
$$= \int_{k_1}^{\infty} \int_{k_2}^{\infty} \int_{k_3}^{\infty} e^{-rT} \left(e^{s_1} - e^{k_1}\right)\left(e^{s_2} - e^{k_2}\right)\left(e^{s_3} - e^{k_3}\right) q_T(s_1, s_2, s_3) ds_3 ds_2 ds_1. \quad (47)$$

The characteristic function corresponding to the joint density is given by

$$\varphi(u_1, u_2, u_3) = E^Q\left(e^{\sum_j iu_j s_j(T)}\right), \quad j = 1, 2, 3. \quad (48)$$

$$\varphi(u_1, u_2, u_3) = \int_{-\infty}^{\infty} e^{iu_1 s_1 + iu_2 s_2 + iu_3 s_3} q_T(s_1, s_2, s_3) ds_3 ds_2 ds_1. \quad (49)$$

Multiplying the right side of Equation (47) and decaying term say α over k_1, k_2, k_3 so as to have square integrable of the right side of (47).

We expressed the result as

$$c_T(k_1, k_2, k_3) = e^{\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3} \times V_T(k_1, k_2, k_3), \quad \alpha_1, \alpha_2, \alpha_3 > 0. \quad (50)$$

For the call option, the Fourier transform of the modified option prices is given by

$$\psi_T(u_1, u_2, u_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(u_1 k_1 + u_2 k_2 + u_3 k_3)} c_T(k_1, k_2, k_3) dk_3 dk_2 dk_1 \quad (51)$$

$$\psi_T(u_1, u_2, u_3) = \int_{IR^3} e^{(\alpha_1 + iu_1)k_1 + (\alpha_2 + iu_2)k_2 + (\alpha_3 + iu_3)k_3}$$
$$\times \int_{k_3}^{+\infty} \int_{k_2}^{+\infty} \int_{k_1}^{+\infty} e^{-rT}\left(e^{s_1} - e^{k_1}\right)\left(e^{s_2} - e^{k_2}\right)\left(e^{s_3} - e^{k_3}\right) \quad (52)$$
$$\times q_T(s_1, s_2, s_3) dk_3 dk_2 dk_1 ds_3 ds_2 ds_1$$

Further simplification yields

$$\psi_T\left(u_1,u_2,u_3\right)=\int_{IR^3}e^{-rT}q_T\left(s_1,s_2,s_3\right)$$

$$\times\int_\infty^{s_3}\int_\infty^{s_2}\int_\infty^{s_1}e^{-rT}\left(\left(e^{s_1}-e^{k_1}\right)\left(e^{s_2}-e^{k_2}\right)\left(e^{s_3}-e^{k_3}\right)\right) \qquad (53)$$

$$\times e^{(\alpha_1+iu_1)k_1+(\alpha_2+iu_2)k_2+(\alpha_3+iu_3)k_3}\,dk_3dk_2dk_1ds_3ds_2ds_1$$

which reduces to

$$\psi_T\left(u_1,u_2,u_3\right)=\int_{IR^3}e^{-rT}q_T\left(s_1,s_2,s_3\right)$$

$$\times\left[\frac{\left(e^{(1+\alpha_1+iu_1)s_1+(1+\alpha_2+iu_2)s_2+(1+\alpha_3+iu_3)s_3}\right)}{(\alpha_1+iu_1)(1+\alpha_1+iu_1)(\alpha_2+iu_2)(1+\alpha_2+iu_2)(\alpha_3+iu_3)(1+\alpha_3+iu_3)}\right]ds_3ds_2ds_1 \qquad (54)$$

Consider a known characteristic function φ_T for the distributions, an analytical close form formula for $\Psi_T\left(u_1,u_2,u_3\right)$ which is the Fourier transform of the 3-dimensional multi-assets is given by

$$V_T\left(k_1,k_2,k_3\right)$$

$$=\frac{e^{-(\alpha_1k_1+\alpha_2k_2+\alpha_3k_3)}}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(u_1k_1+u_2k_2+u_3k_3)}\psi_T\left(u_1,u_2,u_3\right)du_3du_2du_1. \qquad (55)$$

Rewriting the integrals in series form using trapezoidal rule, equation (55) is transformed to

$$V_T\left(k_1,k_2,k_3\right)$$

$$=\frac{e^{-(\alpha_1k_1+\alpha_2k_2+\alpha_3k_3)}}{(2\pi)^3}\underbrace{\sum_{m_1=1}^{N_1}\sum_{m_2=1}^{N_2}\sum_{m_3=1}^{N_3}e^{-i\left(u_{1,m_1}k_1+u_{2,m_2}k_2\right)}\psi_T\left(u_{1,m_1},u_{2,m_2},u_{3,m_3}\right)h_3h_2h_1}_{\Omega(k_1,k_2,k_3)} \qquad (56)$$

where $h_1h_2h_3$, is the product of the integration steps.

The general form of Fast Fourier Transform (FFT) equation for a single asset has been given in the form

$$H\left(k\right)=\sum_{j=1}^{N-1}e^{-i\frac{2\pi}{N}(j-1)(k-1)}y_j,\ \ 1\le k\le N. \qquad (57)$$

Whence, 3-dimensional version of $y_j, j=1,2,3$, in (57) so as to capture the imaginary part of the complex form of y_j, we have

$$\xi_j\mid j=1,2,3:=\sum_{m_1=1}^{N_1-1}\sum_{m_2=1}^{N_1-1}\sum_{m_3=1}^{N_1-1}e^{-i\left(\frac{2\pi}{N_1}j_1m_1+\frac{2\pi}{N_2}j_2m_2+\frac{2\pi}{N_3}j_3m_3\right)}y_j,\ \ j=1,2,3. \qquad (58)$$

where $m_1=1,2,\cdots,N_1-1;m_2=1,2,\cdots,N_2-1;m_3=1,2,\cdots,N_2-1$, respectively.

The integration steps $h_j, j=1,2,3$, in (56) is uniformly chosen such that

$$h_1=\frac{u_{1,m_1}}{\left(m_1-\dfrac{N}{2}\right)},\ \ h_2=\frac{u_{2,m_2}}{\left(m_2-\dfrac{N}{2}\right)},\ \ h_3=\frac{u_{3,m_3}}{\left(m_3-\dfrac{N}{2}\right)}$$

Considering a uniform grid size of $N\times N$ array and set.

The size of the uniform spacing ζ between the N values of vector k with respect to the integration steps $h_j, j = 1, 2, 3$ and Equation (58) has the following relation

$$\zeta_1 h_1 = \zeta_2 h_2 = \zeta_3 h_3 = \frac{2\pi}{N} \tag{59}$$

The options prices V_T over log strikes from Equation (56) is given by

$$V_T \left(k_{1,p_1}, k_{2,p_2}, k_{3,p_3} \right)$$
$$\approx \frac{e^{-\left(\alpha_1 k_{1,p_1} + \alpha_2 k_{2,p_2} + \alpha_3 k_{3,p_3} \right)}}{(2\pi)^3} \Omega \left(k_{1,p_1}, k_{2,p_2}, k_{3,p_3} \right) h_1 h_2 h_3, \tag{60}$$
$$\text{for } 0 \leq p_1, p_2, p_3 \leq N - 1.$$

where

$$\Omega \left(k_{1,p_1}, k_{2,p_2}, k_{3,p_3} \right)$$
$$= \sum_{m_1=1}^{N_1-1} \sum_{m_2=1}^{N_1-1} \sum_{m_3=1}^{N_1-1} e^{-\frac{2\pi}{N} \left(\left(m_1 - \frac{N}{2} \right)\left(p_1 - \frac{N}{2} \right) + \left(m_2 - \frac{N}{2} \right)\left(p_2 - \frac{N}{2} \right) + \left(m_3 - \frac{N}{2} \right)\left(p_3 - \frac{N}{2} \right) \right)} \psi_T \left(u_1, u_2, u_3 \right)$$

5. Numerical Results in 3-Dimensional Assets Case

5.1. Parameters Specification

Consider call option in European sense on an underlying vector of assets and Strikes prices given as

$$S := \begin{cases} S_1 = 100; \\ S_2 = 105; \\ S_3 = 110. \end{cases} \tag{61}$$

$$K := \begin{cases} K_1 = 80; \\ K_2 = 85; \\ K_3 = 90. \end{cases} \tag{62}$$

respectively.

Taking interest rate $r = 0.04$, dividend rate $d = 0.015$ and exercising time $T = \{0.5, 1\}$ for all the assets. The fast Fourier Transform algorithm was used to generate the results in the table below. Setting fineness of integration grid $N = 2^{12}$ and maintain the same integrability parameter $\alpha := \{\alpha_1 = \alpha_2 = \alpha_3\} = 0.25$. For illustration purpose, we assumed the volatilities are stochastic and randn() function was used to generate 3 array of numbers between 1 and 3 for volatilities from the two sources explained by (18) in each case. The following results in the Tables were obtained.

5.2. Tables of Results

Using the same data for the parameters in **Table 1** except that dividend parameter is set to zero. We obtain **Table 2**.

Table 1. Numerical value for a dividend paying multi-assets call options in 3-dimensions with two sources of volatility.

3-underlying Assets Prices (S)	Strike prices (K)	Volatility $\{\sigma^*, \sigma^{rec}\}$	Exercising Time (year)	European call Options Prices (FFT)	European call Options Prices (BSM)	Output Metrics
$S_1 = 100$	80	0.93001, 0.074916	0.5	36.5090918034085	36.36246996	0.1466
$S_2 = 105$	85	−1.85574, −0.098738	0.5	58.4844133882581	58.84131851	−0.3569
$S_3 = 110$	90	0.89337, 1.139327	0.5	62.3503964521184	63.03609669	−0.6857
$S_1 = 100$	80	0.31079, −0.052239	1	23.8042830392887	23.66008671	0.14419
$S_2 = 105$	85	−1.38470, 1.571944	1	22.5677094491566	22.61602268	−0.0483
$S_3 = 110$	90	0.71091, −0.912938	1	22.9463648747717	23.18429634	−0.2379

Table 2. Numerical value for a non-dividend paying multi-assets call options in 3-dimensions with two sources of volatility.

3-underlying Assets Prices (S)	Strike prices (K)	Volatility $\{\sigma^*, \sigma^{rec}\}$	Exercising Time (year)	European call Options Prices (FFT)	European call Options Prices (BSM)	Output metrics
$S_1 = 100$	80	0.93001, 0.074916	0.5	37.0739574519076	36.92704398	0.1469
$S_2 = 105$	85	−1.85574, −0.098738	0.5	59.1066925306292	59.47219301	−0.3655
$S_3 = 110$	90	0.93001, 0.074916	0.5	55.6155902567383	55.89405735	−0.2784
$S_1 = 100$	80	0.31079, −0.052239	1	25.0974529765726	24.95267554	0.1448
$S_2 = 105$	85	−1.38470, 1.571944	1	24.0013372611456	24.05195616	−0.0506
$S_3 = 110$	90	0.71091, −0.912938	1	24.4073616767388	24.65044343	−0.2431

6. Conclusion

Fast Fourier Transform algorithm for valuation of multi-assets in both finite and infinite dimension was presented in this research work. The notion of economic recession induced volatility uncertainty was introduced in our formulation. An intuition behind the introduction of recession induced volatility uncertainty is revealed by huge volatility variation during the period of Economic recession compared to the period of normalcy (recession-free). Nigeria economic recession outbreak in 2016 and its effects on the payoffs uncertainty of Nigeria Stocks Exchange (NSE) among other invetments were among the motivating factors for proposing economic recession induced volatility formulation in Options pricing. This work generalizes the Fast Fourier transform algorithm of Carr & Madan (2009) [6] in the valuation of European call options on a single underlying asset. The algorithm is extended further to include some other steps, then applied to value multi-assets European options in finite and infinite dimensions. Numerical experiment was presented taking Black Scholes model as a bench mark. It is believed that this research work will enhance better understanding of Fast Fourier Transform valuation of multi-assets options in multi-dimensions. We gave the Options output metrics between the two methods. The result obtained is still comparable with little errors hinged on parameters uncertainties.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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