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On the Dynamics of the Solutions of the Rational Recursive Sequences

M. A. El-Moneam[1*]

[1] *Mathematics Department, Faculty of Science and Arts in Farasan, Jazan University, Kingdom of Saudi Arabia.*

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Abstract

In this article, we study the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_n x_{n-k} x_{n-l}}{dx_{n-k} - ex_{n-l}}, \qquad n = 0, 1, 2, ...$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma},..., x_{-l},..., x_{-k}, ..., x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$. Some numerical examples will be given to illustrate our results.

Keywords: Difference equations; prime period two solution; boundedness character; locally asymptotically stable; global attractor; global stability.

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**Corresponding author: E-mail: mabdelmeneam2014@yahoo.com*

1 Introduction

Various natural and social processes can be represented as continuous models, which are in fact a differential equation or a system of differential equations. On the other hand, today more and more attention in many mathematical research centers is devoted to the study of discrete models derived from physics, electrical engineering, mechanical engineering, mathematical biology, epidemiology, economics or social sciences. The reason is that in many situations they adequately describe the process of decision. These models are represented by difference equations or systems of difference equations. Namely, in the mathematical modeling in biology, for example, a system of difference equations is the way in which the two discrete populations (which are often in competition or cooperation) reproduced from one generation to the next (Allee effects in fish, Lotka-Volteraovi discrete models, discrete model type "Hunter-victim" models of type Lesllie-Gower, etc.). Of particular significance in recent times has been testing the models that are represented by rational two-dimensional systems (non-linear) of difference equations, i.e. rational difference equations with quadratic members. There is, first of all, thought of competitive systems (i.e. systems that are modeling the competition between the two populations, companies or similar), and cooperative systems (which is modeling cooperation), and the anti-competitive but also to the other systems of this class. In mathematical terms these are rational systems of difference equations in the plane. For example, in the epidemiology discrete models are describing the interdependence between population suspicious and infected populations. Difference equations proved to be effective in modelling and analysing the discrete dynamical system that arise in signal processing, populations dynamics, heath sciences, economics, and so on. These equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. (see [1]). For an introduction to the general theory of difference equations, we refer the readers to Agarwal [2], Elaydi [3], and Kelley and Peterson [4]. Rational difference equations are important classes of difference equations, where they have many applications in the real life, for example in optics and mathematical biology (see [5]). Recently, there has been an interesting interest in the study of the global behavior of rational difference equations, for example, see [6], [7], [8]–[13]. We believe that the behavior of solutions of rational difference equations provides prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one (see [14]). Recently, [15], Zayed and El-Moneam [16], [17]–[20] have studied the following nonlinear rational difference equations:

$$x_{n+1} = ax_n - \frac{bx_n}{(cx_n - dx_{n-1})}, \qquad n = 0, 1, 2, ... \tag{1.1}$$

$$x_{n+1} = ax_n - \frac{bx_n}{(cx_n - dx_{n-k})}, \qquad n = 0, 1, 2, ... \tag{1.2}$$

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-k}}{(cx_n \pm dx_{n-k})}, \qquad n = 0, 1, 2, ... \tag{1.3}$$

$$x_{n+1} = ax_n - \frac{bx_{n-k}}{(cx_n - dx_{n-k})}, \qquad n = 0, 1, 2, ... \tag{1.4}$$

$$x_{n+1} = ax_{n-k} - \frac{bx_n}{(cx_n - dx_{n-k})}, \qquad n = 0, 1, 2, ... \tag{1.5}$$

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{\beta x_n + \gamma x_{n-k}}{(Cx_n + Dx_{n-k})}, \qquad n = 0, 1, 2, ... \tag{1.6}$$

where in these equations, the parameters and the initial conditions are positive real numbers, while k, l are positive integers such that $k < l$.

The objective of this article is to extend the work of [6], Zayed and El-Moneam (1.1) – (1.6) and investigate some qualitative behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_n x_{n-k} x_{n-l}}{dx_{n-k} - ex_{n-l}}, \qquad n = 0, 1, 2, ... \tag{1.7}$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}$,..., x_{-l},..., $x_{-k}, ..., x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$. That being said, the remainder of this paper is organized as follows: In Section 2, we present some well-know definitions and results that are needed in the sections to follow. In Section 3, using elementary mathematics and nontrivial combinations of ideas, we establish our main results.

Definition 1.1. Consider a difference equation in the form

$$x_{n+1} = F(x_n, x_{n-k}, x_{n-l}, x_{n-\sigma}), \quad n = 0, 1, 2, ... \tag{1.8}$$

where F is a continuous function, while k and l are positive integers such that $k < l < \sigma$. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, \widetilde{x}, \widetilde{x})$. That is, the constant sequence $\{x_n\}$ with $x_n = \widetilde{x}\ for\ all\ \ n \geq -k \geq -l \geq \sigma$ is a solution of that equation.

Definition 1.2. Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of Eq.(1.8). Then we have

(i) An equilibrium point $\widetilde{x}$ of Eq.(1.8) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-\sigma}$, ...,x_{-l}, ...,x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ with $|x_{-\sigma} - \widetilde{x}| + ... + |x_{-l} - \widetilde{x}| + ... + |x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k \geq -l$.

(ii) An equilibrium point $\widetilde{x}$ of Eq.(1.8) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-\sigma}$, ...,x_{-l}, ...,x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ with $|x_{-\sigma} - \widetilde{x}| + ... + |x_{-l} - \widetilde{x}| + ... + |x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of Eq.(1.8) is called a global attractor if for every $x_{-\sigma}$, ...,x_{-l}, ...,x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of Eq.(1.8) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of Eq.(1.8) is called unstable if it is not locally stable.

Definition 1.3. A sequence $\{x_n\}_{n=-\sigma}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geq -\sigma$. A sequence $\{x_n\}_{n=-\sigma}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.

Definition 1.4. Eq.(1.8) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-\sigma}$, ...,x_{-l}, ...,x_{-k}, ..., x_{-1}, $x_0 \in (0, \infty)$ there exists a positive integer N which depends on these initial conditions such that

$$m \leq x_n \leq M \quad for\ all \quad n \geq N.$$

Definition 1.5. The linearized equation of Eq.(1.8) about the equilibrium point $\widetilde{x}$ is defined by the equation

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-k} + \rho_2 z_{n-l} + \rho_3 z_{n-\sigma} = 0, \tag{1.9}$$

where

$$\rho_0 = \frac{\partial F(\widetilde{x}, \widetilde{x}, \widetilde{x}, \widetilde{x})}{\partial x_n}, \quad \rho_1 = \frac{\partial F(\widetilde{x}, \widetilde{x}, \widetilde{x}, \widetilde{x})}{\partial x_{n-k}}, \quad \rho_2 = \frac{\partial F(\widetilde{x}, \widetilde{x}, \widetilde{x}, \widetilde{x})}{\partial x_{n-l}}, \quad \rho_3 = \frac{\partial F(\widetilde{x}, \widetilde{x}, \widetilde{x}, \widetilde{x})}{\partial x_{n-\sigma}}.$$

The characteristic equation associated with Eq.(1.9) is

$$\rho(\lambda) = \lambda^{\sigma+1} - \rho_0 \lambda^{\sigma} - \rho_1 \lambda^{\sigma-k} - \rho_2 \lambda^{\sigma-l} - \rho_3 = 0. \tag{1.10}$$

Theorem 1.1. *[1]. Assume that F is a C^1- function and let $\widetilde{x}$ be an equilibrium point of Eq.(1.8). Then the following statements are true.*
(i) If all roots of Eq.(1.10) lie in the open unit disk $|\lambda| < 1,$ then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.
(ii) If at least one root of Eq.(1.10) has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.
(iii) If all roots of Eq.(1.10) have absolute value greater than one, then the equilibrium point $\widetilde{x}$ is a source.

Theorem 1.2. *[14]. Assume that ρ_0, ρ_1, ρ_2 and $\rho_3 \in R.$ Then*

$$|\rho_0| + |\rho_1| + |\rho_2| + |\rho_3| < 1, \tag{1.11}$$

is a sufficient condition for the asymptotic stability of Eq.(1.8).

Theorem 1.3. *[1]. Consider the difference equation (1.8). Let $\widetilde{x} \in I$ be an equilibrium point of Eq.(1.8). Suppose also that*
(i) F is a nondecreasing function in each of its arguments.
(ii) The function F satisfies the negative feedback property

$$[F(x,x,x,x) - x](x - \widetilde{x}) < 0 \quad for\ all \quad x \in I - \{\widetilde{x}\},$$

where I is an open interval of real numbers. Then $\widetilde{x}$ is global attractor for all solutions of Eq.(1.8).

2 The Local Stability of the Solutions

The equilibrium point $\widetilde{x}$ of Eq.(1.7) is the positive solution of the equation

$$\widetilde{x} = (A + B + C + D)\,\widetilde{x} + \frac{b\,\widetilde{x}^3}{(d-e)\,\widetilde{x}}, \quad d \neq e.$$

If $d \neq e,\ A + B + C + D \neq 1$, then the nonzero equilibrium point $\widetilde{x}$ of Eq.(1.7) is given by

$$\widetilde{x} = \frac{(d-e)}{b}\left[1 - (A + B + C + D)\right]. \tag{2.1}$$

Let us now introduce a continuous function $F : (0,\infty)^4 \longrightarrow (0,\infty)$ which is defined by

$$F(u_0,u_1,u_2,u_3) = Au_0 + Bu_1 + Cu_2 + Du_3 + \frac{bu_0u_1u_2}{du_1 - eu_2}. \tag{2.2}$$

Consequently, we get

$$\begin{cases} \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_0} = 1 - (B + C + D) = \rho_0, \\ \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_1} = B - \frac{e}{(d-e)}\left[1 - (A + B + C + D)\right] = \rho_1, \\ \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_2} = C + \frac{d}{(d-e)}\left[1 - (A + B + C + D)\right] = \rho_2, \\ \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_3} = D = \rho_3. \end{cases} \tag{2.3}$$

Therefore, the linearized equation of Eq.(1.7) about the non zero equilibrium (2.1) is given by

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-k} - \rho_2 z_{n-l} - \rho_3 z_{n-\sigma} = 0, \tag{2.4}$$

where ρ_0, ρ_1, ρ_2 and ρ_3 are given by (2.3).

Theorem 2.1. *If* $A+B+C+D \neq 1$ *and*

$$\begin{aligned} &|(d-e)[1-(B+C+D)]| + |B(d-e) - e[1-(A+B+C+D)]| \\ &+ |C(d-e) + d[1-(A+B+C+D)]| + D < |d-e|, \end{aligned} \tag{2.5}$$

then, the nonzero equilibrium point (2.1) is locally asymptotically stable.

Proof. The result follows directly from (1.10) and (2.5), and then apply Theorem 1.2. Thus, the proof is now completed. □

3 Periodic Solutions

In this section, we study the existence of periodic solutions of Eq.(1.7). The following theorem states the necessary and sufficient conditions that the equation (1.7) has periodic solutions of prime period two.

Theorem 3.1. *(i) If* k, l *and* σ *are both even positive integers, or*
(ii) If k, l *are even and* σ *is odd positive integers.*
Then, the following statements are true:

(1) If $d > e$, *Eq.(1.7) has no prime period two solution.*

(2) If $d < e$, *Eq.(1.7) has prime period two solution if*

$$A+B+C+D > 3. \tag{3.1}$$

Proof. Assume that there exists distinct positive solutions

$$\ldots\ldots., P, Q, P, Q, \ldots\ldots.. \tag{3.2}$$

of prime period two of Eq.(1.7).

(i) If k, l and σ are all even positive integers, then $x_n = x_{n-k} = x_{n-l} = x_{n-\sigma}$. It follows from Eq.(1.7) that

$$P = (A+B+C+D)\,Q + \frac{bQ^2}{(d-e)}, \tag{3.3}$$

and

$$Q = (A+B+C+D)\,P + \frac{bP^2}{(d-e)}, \tag{3.4}$$

where $d \neq e$. Consequently, we get

$$(d-e)\,P = (d-e)\,(A+B+C+D)\,Q + bQ^2, \tag{3.5}$$

and

$$(d-e)\,Q = (d-e)\,(A+B+C+D)\,P + bP^2. \tag{3.6}$$

By subtracting (3.6) from (3.5), we get

$$P+Q = -\frac{(d-e)}{b}\left[1+(A+B+C+D)\right]. \tag{3.7}$$

If $d > e$, we deduce from (3.7) that $P+Q < 0$. This is a contradiction. Thus, Eq.(1.7) has no prime period two solution. The proof of part (1) of (i) follows. If $d < e$, we deduce from (3.7) that $P+Q > 0$, which is always true. Consequently, we add Eqs.(3.5) and (3.6) and using (3.7) and we have

$$PQ = \left(\frac{d-e}{b}\right)^2 \left[1+(A+B+C+D)\right]. \tag{3.8}$$

Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2 - (P+Q)\,t + PQ = 0. \tag{3.9}$$

Thus, we deduce that

$$(P+Q)^2 > 4PQ. \tag{3.10}$$

Substituting (3.7) and (3.8) into (3.10), we get the condition (3.1), and then the proof of part (2) of (i) follows. Similarly, we can prove the theorem, in case (ii) which is omitted here and the proof is now completed. □

Theorem 3.2. *(i) If k,l and σ are both odd positive integers and $d \neq e$, $A+1 \neq B+C+D$, or*
(ii) If σ is even and k,l are odd positive integers and $d \neq e$, $(A+D+1) \neq (B+C)$.
Then, Eq.(1.7) has no prime period two solution.

Proof. Following the proof of Theorem 3.1, we deduce that:

(i) If k,l and σ are both odd positive integers, then $x_{n+1} = x_{n-k} = x_{n-l} = x_{n-\sigma}$. It follows from Eq.(1.7) that

$$P = AQ + (B+C+D)\,P + \frac{bPQ}{(d-e)}, \tag{3.11}$$

and

$$Q = AP + (B+C+D)\,Q + \frac{bPQ}{(d-e)}. \tag{3.12}$$

Consequently, we get

$$(d-e)\,P = A\,(d-e)\,Q + (d-e)\,(B+C+D)\,P + bPQ, \tag{3.13}$$

and

$$(d-e)\,Q = A\,(d-e)\,P + (d-e)\,(B+C+D)\,Q + bPQ. \tag{3.14}$$

By subtracting (3.14) from (3.13), we get

$$(P-Q)\,[(A+1) - (B+C+D)] = 0.$$

Since $(d-e) \neq 0$ and $[(A+1) - (B+C+D)] \neq 0$, then $P = Q$. This is a contradiction, and then the proof of part (i) follows.

(ii) If σ is even and k,l are odd positive integers, then $x_n = x_{n-\sigma}$ and $x_{n+1} = x_{n-k} = x_{n-l}$. It follows from Eq.(1.7) that

$$P = (A+D)\,Q + (B+C)\,P + \frac{bPQ}{(d-e)}, \tag{3.15}$$

and

$$Q = (A+D)\,P + (B+C)\,Q + \frac{bPQ}{(d-e)}. \tag{3.16}$$

Consequently, we get

$$(d-e)\,P = (d-e)\,(A+D)\,Q + (d-e)\,(B+C)\,P + bPQ, \tag{3.17}$$

and

$$(d-e)\,Q = (d-e)\,(A+D)\,P + (d-e)\,(B+C)\,Q + bPQ. \tag{3.18}$$

By subtracting (3.18) from (3.17), we get

$$(P-Q)\,(d-e)\,[(B+C) - (A+D+1)] = 0.$$

Since $(d-e) \neq 0$ and $[(B+C) - (A+D+1)] \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is now completed. □

4 Boundedness of the Solutions

In this section, we investigate the boundedness of the positive solutions of Eq.(1.7).

Theorem 4.1. *Let $\{x_n\}_{n=-\sigma}^{\infty}$ be a solution of Eq.(1.7). Then the following statements are true:*

(i) Suppose $b < d$ and for some $N \geq 0,$ $the\ intial\ conditions$

$$x_{N-\sigma+1},...,x_{N-l+1},...,x_{N-k+1},...,x_{N-1},x_N \in \left[\frac{b}{d},1\right],$$

are valid, then for $b \neq e,\ d^2 \neq be$ we have the inequality

$$\frac{b}{d}\left[(A+B+C+D)+\frac{b^2}{(d^2-be)}\right] \leq x_n \leq (A+B+C+D)+\frac{b}{(b-e)}, \quad for\ \ all \quad n \geq N. \tag{4.1}$$

(ii) Suppose $b > d$ and for some $N \geq 0,$ $the\ intial\ conditions$

$$x_{N-\sigma+1},...,x_{N-l+1},...,x_{N-k+1},...,x_{N-1},x_N \in \left[1,\frac{b}{d}\right],$$

are valid, then for $b \neq e,\ d^2 \neq be$ we have the inequality

$$(A+B+C+D)+\frac{b}{(b-e)} \leq x_n \leq \frac{b}{d}\left[(A+B+C+D)+\frac{b^2}{(d^2-be)}\right], \quad for\ \ all \quad n \geq N. \tag{4.2}$$

Proof. First of all, if for some $N \geq 0$ and $\frac{b}{d} \leq x_N \leq 1$ and $b \neq e,$ we have

$$\begin{aligned} x_{N+1} &= Ax_N + Bx_{N-k} + Cx_{N-l} + Dx_{N-\sigma} + \frac{bx_N x_{N-k} x_{N-l}}{(dx_{N-k} - ex_{N-l})} \\ &\leq (A+B+C+D) + \frac{b}{(dx_{N-k} - ex_{N-l})}. \end{aligned} \tag{4.3}$$

It is easy to see that $dx_{N-k} - ex_{N-l} \geq b-e,$ then for $b \neq e,$ we get

$$x_{N+1} \leq (A+B+C+D)+\frac{b}{(b-e)}. \tag{4.4}$$

Similarly, we can show that

$$x_{N+1} \geq \frac{b}{d}\left[(A+B+C+D)+\frac{b^2}{d\ (dx_{N-k}-ex_{N-l})}\right]. \tag{4.5}$$

But, one can show that $dx_{N-k} - ex_{N-l} \geq \frac{1}{d}\left(d^2 - be\right)$, then for $d^2 \neq be,$ we get

$$x_{N+1} \geq \frac{b}{d}\left[(A+B+C+D)+\frac{b^2}{(d^2-be)}\right]. \tag{4.6}$$

From (4.4) and (4.6) we deduce for all $n \geq N$ that the inequality (4.1) is valid. Hence the proof of part (i) is completed. Similarly, if $1 \leq x_N \leq \frac{b}{d}$, then we can prove part (ii) which is omitted here. Thus, the proof is now completed. □

5 Global Stability

In this section we study the global asymptotic stability of the positive solutions of Eq.(1.7).

Theorem 5.1. *The nonzero equilibrium point (2.1) of Eq.(1.7) is global attractor.*

Proof. We consider the following function:

$$F(x,y,z,w) = Ax + By + Cz + Dw + \frac{bxyz}{(dy-ez)}. \tag{5.1}$$

where $dy \neq ez,$ provided that $\left[A\left(dy-ez\right)^2 + byz\left(dy-ez\right)\right] > 0,\ B\left(dy-ez\right)^2 > bexz^2$ and $\left[C\left(dy-ez\right)^2 + bdxy^2\right] > 0$. It is easy to verify the condition (i) of Theorem 1.3. Let us now verify the condition (ii) of Theorem 1.3 as follows:

$$\begin{aligned}
\left[F(x,x,x,x)-x\right](x-\widetilde{x}) &= \left\{(A+B+C+D)\,x + \frac{bx^2}{(d-e)} - x\right\} \times \\
&\quad \left\{x - \frac{(d-e)}{b}\left[1-(A+B+C+D)\right]\right\} \\
&= \frac{bx}{(d-e)}\left\{x - \frac{(d-e)}{b}\left[1-(A+B+C+D)\right]\right\}^2 < 0,
\end{aligned} \tag{5.2}$$

which is valid for all x satisfying the inequality

$$\frac{bx}{(d-e)} < 0. \tag{5.3}$$

According to Theorem 1.3, the nonzero equilibrium point $\widetilde{x}$ given by (2.1) is global attractor if the condition (5.3) is valid. Thus, the proof is now completed. □

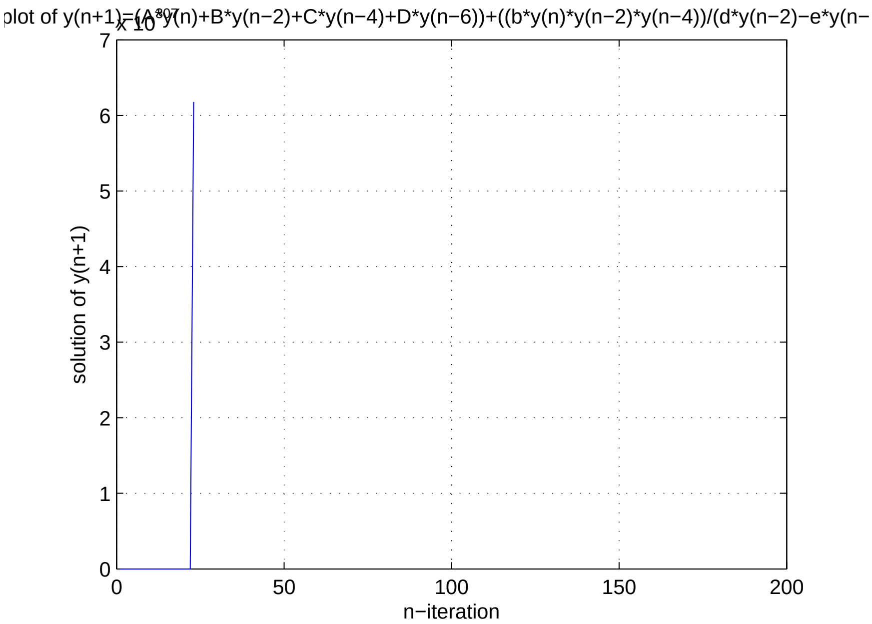
On combining the two Theorems 2.1 and 5.1, we have the following result:

Theorem 5.2. *The nonzero quilibrium point (2.1) of Eq.(1.7) is globally asymptotically stable.*

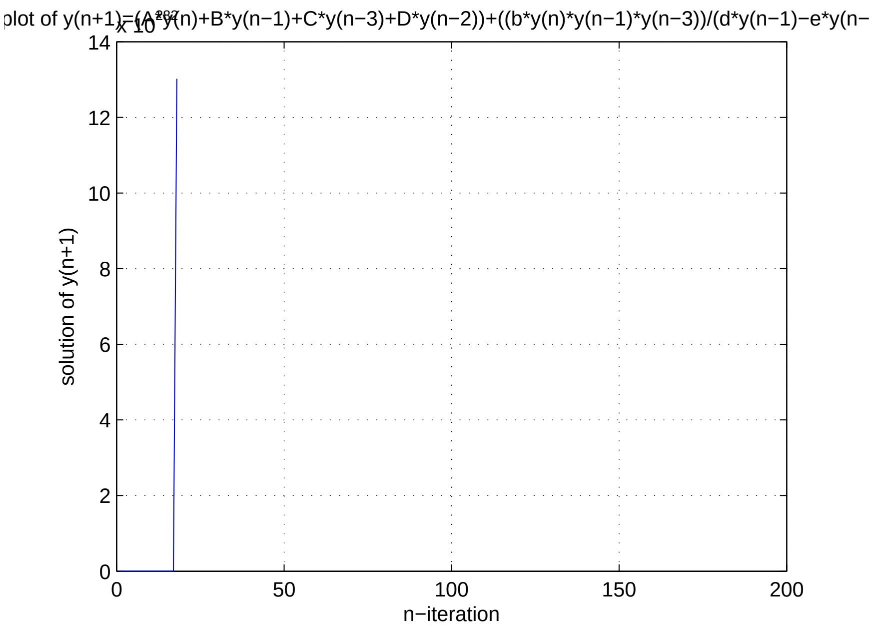
6 Numerical Examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq.(1.7).

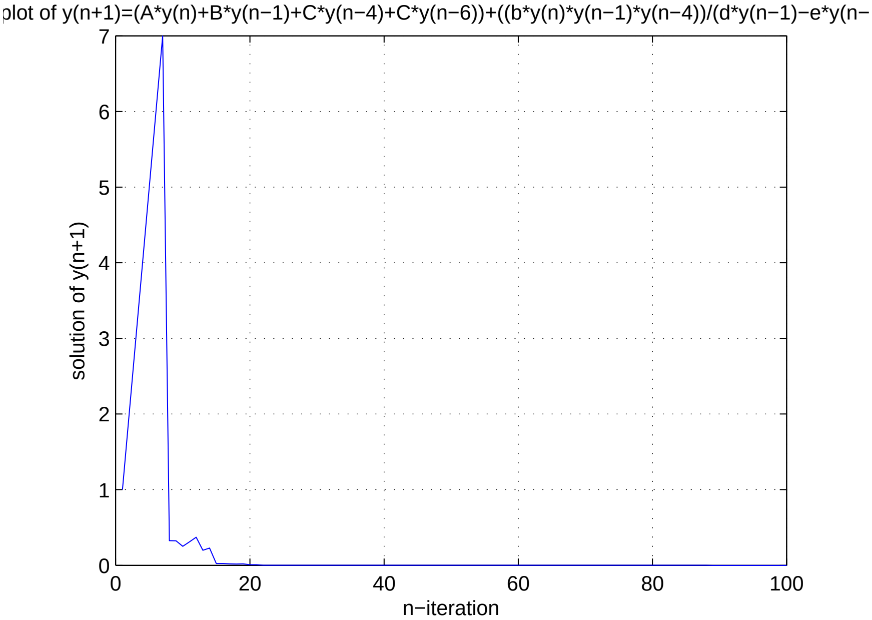
Example 1. Figure 1, shows that Eq.(1.7) has no prime period two solutions if $k = 2,\ l = 4,\ \sigma = 6,\ x_{-6} = 1,\ x_{-5} = 2, x_{-4} = 3,\ x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6,\ x_0 = 7,\ A = 300,\ B = 200,\ C = 100,\ D = 75,\ b = 50,\ d = 30,\ e = 20.$



Example 2. Figure 2, shows that Eq.(1.7) has no prime period two solutions if $k = 1,\ l = 3,\ \sigma = 2,\ x_{-3} = 1,\ x_{-2} = 2, x_{-1} = 3,\ x_0 = 4,\ \ A = 5000,\ B = 1500,\ C = 1000,\ D = 750,\ b = 500,\ d = 300,\ e = 200.$



Example 3. Figure 3, shows that Eq.(1.7) is globally asymptotically stable when ($\widetilde{x} \neq 0$) if $k = 1,\ l = 4,\ \sigma = 6,\ x_{-6} = 1,\ x_{-5} = 2,\ x_{-4} = 3,\ x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6,\ x_0 = 7,\ \ A = 0.01,\ B = 0.02,\ C = 0.03,\ D = 0.04,\ b = 0.4,\ d = 500,\ e = 5.$



7 Conclusions

We have discussed some properties of the nonlinear rational difference equation (1.7), namely the periodicity, the boundedness and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions. Our results in this article can be considered as a more generalization than the results obtained in Refs.[15–20]. Note that example 1 verifies Theorem 3.1 (part (1) of (i)) which shows that if k, l and σ are all even positive integers, then Eq.(1.7) has no prime period two solution and example 2 verifies Theorem 3.2 (ii) which shows that if σ is even and k, l are odd positive integers, then Eq.(1.7) has no prime period two solution. But example 3 verifies Theorem 5.2 which shows that Eq.(1.7) is globally asymptotically stable when ($\widetilde{x} \neq 0$).

Competing Interests

The author declares that no competing interests exist.

References

[1] Grove EA, Ladas G. Periodicities in nonlinear difference equations, Vol.4, Chapman & Hall / CRC; 2005.

[2] Agarwal R. Difference Equations and Inequalities. Theory, Methods and Applications. Marcel Dekker Inc, New York, Second Edition. 2000;228.

[3] Elaydi SN. An introduction to difference equations, Springer, New York; 1999.

[4] Kelley WG, Peterson AC. Difference equations: An introduction with applications, Harcour Academic, New York; 2001.

[5] Saary TI. Modern nonlinear equations, Mc Graw Hill, New York; 1967.

[6] El-Moneam MA, Alamoudy SO. On study of the asymptotic behavior of some rational difference equations. DCDIS Series A: Mathematical Analysis. 2014;21:89-109.

[7] El-Moneam MA, Zayed EME. Dynamics of the rational difference equation. Inf. Sci. Lett. 2014;3(2):45-53.

[8] Zayed EME, El-Moneam MA. On the rational recursive sequence $x_{n+1} = (A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}) / (B + \beta_0 x_n + \beta_1 x_{n-\tau})$, Acta Math. Vietnamica. 2011;36:73-87.

[9] Zayed EME, El-Moneam MA. On the rational recursive sequence $x_{n+1} = (\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}) / (\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k})$, Math. Bohemica. 2010;135:319-336.

[10] Zayed EME, El-Moneam MA. On the global asymptotic stability for a rational recursive sequence, Iranian Journal of science and Technology (A: siences). 2011;35(A4):333-339.

[11] Zayed EME, El-Moneam MA. On the rational recursive sequence $x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-m} + \alpha_3 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-m} + \beta_3 x_{n-k}}$, WSEAS Transactions Math. 2012;11:373-382.

[12] Zayed EME, El-Moneam MA. On the qualitative study of the nonlinear difference equation $x_{n+1} = \frac{\alpha x_{n-\sigma}}{\beta + \gamma x_{n-\tau}^p}$. Fasciculi Mathematici. 2013;50:137-147.

[13] Zayed EME, El-Moneam MA. Dynamics of the rational difference equation $x_{n+1} = \gamma x_n + \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}}$, Comm. Appl. Nonl. Anal. 2014;21:43-53.

[14] Kulenovic MRS, Ladas G. Dynamics of second order rational difference equations with open problems and conjectures, Chapman & Hall / CRC, Florida; 2002.

[15] Elabbasy EM, El- Metwally H, Elsayed EM. On the difference equation $x_{n+1} = a x_n - b x_n / (c x_n - d x_{n-1})$, Advances in Difference Equations. Article ID 82579. 2006;1-10. doi: 10.1155/2006/82579.

[16] Zayed EME, El-Moneam MA. On the rational recursive sequence $x_{n+1} = a x_n - b x_n / (c x_n - d x_{n-k})$. Comm. Appl. Nonlinear Analysis. 2008;15:47-57.

[17] Zayed EME, El-Moneam MA. On the Rational Recursive Sequence $x_{n+1} = \gamma x_{n-k} + (a x_n + b x_{n-k}) / (c x_n - d x_{n-k})$. Bull. Iranian Math. Soc. 2010;36:103-115.

[18] Zayed EME, El-Moneam MA. On the rational recursive sequence $x_{n+1} = A x_n + B x_{n-k} + (\beta x_n + \gamma x_{n-k}) / (C x_n + D x_{n-k})$. Acta Appl. Math. 2010;111:287-301.

[19] Zayed EME, El-Moneam MA. On the rational recursive two sequences $x_{n+1} = a x_{n-k} + b x_{n-k} / (c x_n + \delta d x_{n-k})$. Acta Math. Vietnamica. 2010;35:355-369.

[20] Zayed EME, El-Moneam MA. On the global attractivity of two nonlinear difference equations, J. Math. Sci. 2011;177:487-499.

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