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On the sum of the cubes of generalized balancing numbers: The sum formula $\sum_{k=0}^{n} x^k W_{mk+j}^3$

Yüksel Soykan[1], Erkan Taşdemir[2,*] and Can Murat Dikmen[1]

[1] Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.

[2] Pınarhisar Vocational School, Kırklareli University, 39300, Kırklareli, Turkey.

* Correspondence: erkantasdemir@hotmail.com

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^3$ for generalized balancing numbers are presented. As special cases, we give sum formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers.

Keywords: Balancing numbers; Modified Lucas-balancing numbers; Lucas-balancing numbers; Sum formulas.

MSC: 11B37; 11B39; 11B83.

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1. Introduction

Behera and Panda [1] defined balancing numbers n as solutions of the diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some natural number r, called the balancer corresponding to n. The nth balancing number is denoted by B_n. Moreover, $C_n = \sqrt{8B_n^2 + 1}$ is called the nth Lucas-balancing number (see [2]). In fact, B_n and C_n satisfy the second order linear recurrence relations

$$B_n = 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1,$$

and

$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3$$

respectively. $(B_n)_{n\geq 0}$ is the sequence $A001109$ in the OEIS [3], whereas $(B_n)_{n\geq 0}$ is the id-number $A001541$ in OEIS. Balancing and Lucas-balancing sequences has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,4–27].

A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2}, \tag{1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1) holds for all integer n.

The Binet formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n,$$

where α and β are the roots of the quadratic equation $x^2-6x+1=0$. Moreover

$$\begin{aligned}\alpha &= 3+2\sqrt{2},\\ \beta &= 3-2\sqrt{2}.\end{aligned}$$

Note that

$$\begin{aligned}\alpha+\beta &= 6,\\ \alpha\beta &= 1,\\ \alpha-\beta &= 4\sqrt{2}.\end{aligned}$$

Now we define three special cases of the sequence $\{W_n\}$. Balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations,

$$\begin{aligned}B_n &= 6B_{n-1}-B_{n-2}, \quad B_0=0, B_1=1, &(2)\\ H_n &= 6H_{n-1}-H_{n-2}, \quad H_0=2, H_1=6, &(3)\\ C_n &= 6C_{n-1}-C_{n-2}, \quad C_0=1, C_1=3. &(4)\end{aligned}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned}B_{-n} &= 6B_{-(n-1)}-B_{-(n-2)},\\ H_{-n} &= 6H_{-(n-1)}-H_{-(n-2)},\\ C_{-n} &= 6C_{-(n-1)}-C_{-(n-2)},\end{aligned}$$

for $n=1,2,3,\ldots$ respectively. Therefore, recurrences (2)-(4) hold for all integer n. For more information on generalized balancing numbers, see Soykan [28].

2. The sum formula $\sum_{k=0}^{n} x^k W_{mk+j}^3$

The following theorem presents sum formulas of generalized balancing numbers;

Theorem 1. *Let x be a real (or complex) number. For all integers m and j, for generalized balancing numbers (the case $r=6, s=-1$), we have the following sum formulas:*

(a) *If $(x^2-xH_{3m}+1)(x^2-xH_m+1)\neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_1}{32(x^2-xH_{3m}+1)(x^2-xH_m+1)}, \quad (5)$$

where

$\Psi_1 = 32x^{n+1}(x^2-xH_m+1)W_{mn-m+j}^3 + 32x^{n+1}(x-H_{3m})(x^2-xH_m+1)W_{mn+j}^3 - 32x(x^2-xH_m+1)W_{j-m}^3 + 32(x^2-xH_m+1)W_j^3 + 3x^n x(x^2-xH_{3m}+1)(W_1^2+W_0^2-6W_0W_1)W_{mn+m+j} + 3x^n x(x^2-xH_m+1)(W_1^2+W_0^2-6W_0W_1)W_{mn-m+j} - 3x^n x(x^2H_m-(xH_m-1)H_{3m})(W_1^2+W_0^2-6W_0W_1)W_{mn+j} - 3x(x^2-xH_{3m}+1)(W_1^2+W_0^2-6W_0W_1)W_{m+j} - 3x(x^2-xH_m+1)(W_1^2+W_0^2-6W_0W_1)W_{j-m} + 3x(x^2H_m-H_{3m}(xH_m-1))(W_1^2+W_0^2-6W_0W_1)W_j$.

(b) *If $(x^2-xH_{3m}+1)(x^2-xH_m+1)=u(x-a)(x-b)(x-c)(x-d)=0$ for some $u,a,b,c,d\in\mathbb{C}$ with $u\neq 0$ and $a\neq b\neq c\neq d$, i.e., $x=a$ or $x=b$ or $x=c$ or $x=d$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_2}{\Lambda_1},$$

where

$\Psi_2 = 32x^n(x^2(n+3)-x(n+2)H_m+n+1)W_{mn-m+j}^3 + 32((n+4)x^3-(H_m+H_{3m})(n+3)x^2 + (H_mH_{3m}+1)(n+2)x-(n+1)H_{3m})x^nW_{mn+j}^3 + 32(-3x^2+2xH_m-1)W_{j-m}^3 + 32(2x-H_m)W_j^3 + 3((n+$

$3)x^2 - x(n+2)H_{3m} + n + 1)(W_1^2 + W_0^2 - 6W_0W_1)x^nW_{mn+m+j} + 3((n+3)x^2 - x(n+2)H_m + n + 1)x^n(W_1^2 + W_0^2 - 6W_0W_1)W_{mn-m+j} + 3(-(n+3)x^2H_m + x(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 + W_0^2 - 6W_0W_1)W_{mn+j} + 3(-3x^2 + 2xH_{3m} - 1)(W_1^2 + W_0^2 - 6W_0W_1)W_{m+j} + 3(-3x^2 + 2xH_m - 1)(W_1^2 + W_0^2 - 6W_0W_1)W_{j-m} + 3(3x^2H_m - 2xH_mH_{3m} + H_{3m})(W_1^2 + W_0^2 - 6W_0W_1)W_j,$

and

$\Lambda_1 = 32(4x^3 - 3(H_m + H_{3m})x^2 + 2(2 + H_mH_{3m})x - (H_m + H_{3m})).$

(c) *If* $(x^2 - xH_{3m} + 1)(x^2 - xH_m + 1) = u(x-a)^2(x-b)(x-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$, *then if* $x = b$ *or* $x = c$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_3}{\Lambda_2},$$

where

$\Psi_3 = 32x^n(x^2(n+3) - x(n+2)H_m + n + 1)W_{mn-m+j}^3 + 32((n+4)x^3 - (H_m + H_{3m})(n+3)x^2 + (H_mH_{3m} + 1)(n+2)x - (n+1)H_{3m})x^nW_{mn+j}^3 + 32(-3x^2 + 2xH_m - 1)W_{j-m}^3 + 32(2x - H_m)W_j^3 + 3((n+3)x^2 - x(n+2)H_{3m} + n + 1)(W_1^2 + W_0^2 - 6W_0W_1)x^nW_{mn+m+j} + 3((n+3)x^2 - x(n+2)H_m + n + 1)x^n(W_1^2 + W_0^2 - 6W_0W_1)W_{mn-m+j} + 3(-(n+3)x^2H_m + x(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 + W_0^2 - 6W_0W_1)W_{mn+j} + 3(-3x^2 + 2xH_{3m} - 1)(W_1^2 + W_0^2 - 6W_0W_1)W_{m+j} + 3(-3x^2 + 2xH_m - 1)(W_1^2 + W_0^2 - 6W_0W_1)W_{j-m} + 3(3x^2H_m - 2xH_mH_{3m} + H_{3m})(W_1^2 + W_0^2 - 6W_0W_1)W_j,$

and

$\Lambda_2 = 32(4x^3 - 3(H_m + H_{3m})x^2 + 2(2 + H_mH_{3m})x - (H_m + H_{3m}))$, *and if* $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_4}{64(6x^2 - 3x(H_m + H_{3m}) + 2 + H_mH_{3m})},$$

where

$\Psi_4 = 32((n+3)(n+2)x^2 - x(n+2)(n+1)H_m + n(n+1))x^{n-1}W_{mn-m+j}^3 + 32((n+4)(n+3)x^3 - (n+3)(n+2)(H_m + H_{3m})x^2 + x(n+2)(n+1)(H_mH_{3m} + 1) - n(n+1)H_{3m})x^{n-1}W_{mn+j}^3 + 64(H_m - 3x)W_{j-m}^3 + 64W_j^3 + 3((n+3)(n+2)x^2 - x(n+2)(n+1)H_{3m} + n(n+1))(W_1^2 + W_0^2 - 6W_0W_1)x^{n-1}W_{mn+m+j} + 3x^{n-1}((n+3)(n+2)x^2 - x(n+2)(n+1)H_m + n(n+1))(W_1^2 + W_0^2 - 6W_0W_1)W_{mn-m+j} + 3x^{n-1}(-x^2(n+3)(n+2)H_m + x(n+2)(n+1)H_{3m}H_m - n(n+1)H_{3m})(W_1^2 + W_0^2 - 6W_0W_1)W_{mn+j} + 6(H_{3m} - 3x)(W_1^2 + W_0^2 - 6W_0W_1)W_{m+j} + 6(H_m - 3x)(W_1^2 + W_0^2 - 6W_0W_1)W_{j-m} + 6(3x - H_{3m})H_m(W_1^2 + W_0^2 - 6W_0W_1)W_j.$

(d) *If* $(x^2 - xH_{3m} + 1)(x^2 - xH_m + 1) = u(x-a)^3(x-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $x = a$ *or* $x = b$, *then if* $x = b$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_4}{\Lambda_3},$$

where

$\Psi_5 = 32x^n(x^2(n+3) - x(n+2)H_m + n + 1)W_{mn-m+j}^3 + 32((n+4)x^3 - (H_m + H_{3m})(n+3)x^2 + (H_mH_{3m} + 1)(n+2)x - (n+1)H_{3m})x^nW_{mn+j}^3 + 32(-3x^2 + 2xH_m - 1)W_{j-m}^3 + 32(2x - H_m)W_j^3 + 3((n+3)x^2 - x(n+2)H_{3m} + n + 1)(W_1^2 + W_0^2 - 6W_0W_1)x^nW_{mn+m+j} + 3((n+3)x^2 - x(n+2)H_m + n + 1)x^n(W_1^2 + W_0^2 - 6W_0W_1)W_{mn-m+j} + 3(-(n+3)x^2H_m + x(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 + W_0^2 - 6W_0W_1)W_{mn+j} + 3(-3x^2 + 2xH_{3m} - 1)(W_1^2 + W_0^2 - 6W_0W_1)W_{m+j} + 3(-3x^2 + 2xH_m - 1)(W_1^2 + W_0^2 - 6W_0W_1)W_{j-m} + 3(3x^2H_m - 2xH_mH_{3m} + H_{3m})(W_1^2 + W_0^2 - 6W_0W_1)W_j,$

and

$\Lambda_3 = 32(4x^3 - 3(H_m + H_{3m})x^2 + 2(2 + H_mH_{3m})x - (H_m + H_{3m})),$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_6}{192(4x - H_m - H_{3m})},$$

where

$\Psi_6 = 32(n+1)((n+3)(n+2)x^2 - xn(n+2)H_m + n(n-1))x^{n-2}W_{mn-m+j}^3 + 32((n+3)(n+2)(n+4)x^3 - (n+3)(n+2)(n+1)(H_m + H_{3m})x^2 + n(n+2)(n+1)(H_mH_{3m} + 1)x - n(n-1)(n+$

$1)H_{3m})x^{n-2}W_{mn+j}^{3} - 192W_{j-m}^{3} + 3(n+1)((n+3)(n+2)x^{2} - xn(n+2)H_{3m} + n(n-1))(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})x^{n-2}W_{mn+m+j} + 3(n+1)((n+3)(n+2)x^{2} - xn(n+2)H_{m} + n(n-1))(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})x^{n-2}W_{mn-m+j} + 3(n+1)(-x^{2}(n+3)(n+2)H_{m} + xn(n+2)H_{3m}H_{m} - n(n-1)H_{3m})(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})x^{n-2}W_{mn+j} - 18(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{m+j} - 18(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{j-m} + 18H_{m}(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{j}.$

(e) *If* $(x^{2} - xH_{3m} + 1)(x^{2} - xH_{m} + 1) = u(x-a)^{4} = 0$ *for some* $u, a \in \mathbb{C}, u \neq 0$ *i.e.,* $x = a$ *then*

$$\sum_{k=0}^{n} x^{k}W_{mk+j}^{3} = \frac{\Psi_{7}}{768},$$

where

$\Psi_{7} = 32n(n+1)((n+3)(n+2)x^{2} - x(n-1)(n+2)H_{m} + (n-1)(n-2))x^{n-3}W_{mn-m+j}^{3} + 32(n+1)(x^{3}(n+4)(n+3)(n+2) - x^{2}n(n+3)(n+2)(H_{m} + H_{3m}) + xn(n-1)(n+2)(H_{m}H_{3m} + 1) - n(n-1)(n-2)H_{3m})x^{n-3}W_{mn+j}^{3} + 3n(n+1)(x^{2}(n+3)(n+2) - x(n+2)(n-1)H_{3m} + (n-1)(n-2))(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})x^{n-3}W_{mn+m+j} + 3n(n+1)(x^{2}(n+3)(n+2) - x(n+2)(n-1)H_{m} + (n-1)(n-2))(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})x^{n-3}W_{mn-m+j} + 3n(n+1)(-x^{2}(n+3)(n+2)H_{m} + x(n+2)(n-1)H_{3m}H_{m} - (n-1)(n-2)H_{3m})(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})x^{n-3}W_{mn+j}.$

Proof. Take $r = 6, s = -1$ and $H_{n} = H_{n}$ in Soykan [29], Theorem 2.1.]. □

Note that (5) can be written in the following form:

$$\sum_{k=1}^{n} x^{k}W_{mk+j}^{2} = \frac{\Psi_{8}}{32(x^{2} - xH_{3m} + 1)(x^{2} - xH_{m} + 1)},$$

where

$\Psi_{8} = 32x^{n+1}(x^{2} - xH_{m} + 1)W_{mn-m+j}^{3} + 32x^{n+1}(x - H_{3m})(x^{2} - xH_{m} + 1)W_{mn+j}^{3} - 32x(x^{2} - xH_{m} + 1)W_{j-m}^{3} + 32(H_{3m} - x)(x^{2} - xH_{m} + 1)xW_{j}^{3} + 3x^{n}x(x^{2} - xH_{3m} + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{mn+m+j} + 3x^{n}x(x^{2} - xH_{m} + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{mn-m+j} - 3x^{n}x(x^{2}H_{m} - (xH_{m} - 1)H_{3m})(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{mn+j} - 3x(x^{2} - xH_{3m} + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{m+j} - 3x(x^{2} - xH_{m} + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{j-m} + 3x(x^{2}H_{m} - H_{3m}(xH_{m} - 1))(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{j}.$

As special cases of m and j in the last Theorem, we obtain the following proposition;

Proposition 1. *For generalized balancing numbers (the case* $r = 6, s = -1$*), we have the following sum formulas for* $n \geq 0$*:*

(a) $(m = 1, j = 0)$

If $(x^{2} - 6x + 1)(x^{2} - 198x + 1) \neq 0$*, i.e.,* $x \neq 99 + 70\sqrt{2}$*,* $x \neq 99 - 70\sqrt{2}$*,* $x \neq 3 + 2\sqrt{2}$*,* $x \neq 3 - 2\sqrt{2}$*, then*

$$\sum_{k=0}^{n} x^{k}W_{k}^{3} = \frac{\Psi_{1}}{32(x^{2} - 6x + 1)(x^{2} - 198x + 1)},$$

where

$\Psi_{1} = 32x^{n+1}(x - 198)(x^{2} - 6x + 1)W_{n}^{3} + 32x^{n+1}(x^{2} - 6x + 1)W_{n-1}^{3} + 3x^{n+1}(x^{2} - 198x + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{n+1} - 18x^{n+1}(x^{2} - 198x + 33)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{n} + 3x^{n+1}(x^{2} - 6x + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{n-1} - 32(-x(x^{2} + 12x + 1)W_{1}^{3} + (216x^{3} - 1189x^{2} + 204x - 1)W_{0}^{3} + 18x^{2}(x+6)W_{1}^{2}W_{0} - 18x^{2}(6x+1)W_{0}^{2}W_{1}),$

and

if $(x^{2} - 6x + 1)(x^{2} - 198x + 1) = 0$*, i.e.,* $x = 99 + 70\sqrt{2}$ *or* $x = 99 - 70\sqrt{2}$ *or* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^{k}W_{k}^{3} = \frac{\Psi_{2}}{128(x^{3} - 153x^{2} + 595x - 51)},$$

where

$\Psi_{2} = 32x^{n}(n(x - 198)(x^{2} - 6x + 1) + 4x^{3} - 612x^{2} + 2378x - 198)W_{n}^{3} + 32x^{n}(n(x^{2} - 6x + 1) + 3x^{2} - 12x + 1)W_{n-1}^{3} + 3x^{n}(n(x^{2} - 198x + 1) + 3x^{2} - 396x + 1)(W_{1}^{2} + W_{0}^{2} - 6W_{0}W_{1})W_{n+1} - 18x^{n}(n(x^{2} - 198x +$

$33)+3x^2-396x+33)(W_1^2+W_0^2-6W_0W_1)W_n+3x^n(n(x^2-6x+1)+3x^2-12x+1)(W_1^2+W_0^2-6W_0W_1)W_{n-1}+32((3x^2+24x+1)W_1^3-2(324x^2-1189x+102)W_0^3-54x(x+4)W_1^2W_0+36x(9x+1)W_0^2W_1)$.

(b) $(m=2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^3=\frac{\Psi_1}{32\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x-39202)(x^2-34x+1)W_{2n}^3+32x^{n+1}(x^2-34x+1)W_{2n-2}^3+3x^{n+1}(x^2-39202x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n+2}-102x^{n+1}(x^2-39202x+1153)(W_1^2+W_0^2-6W_0W_1)W_{2n}+3x^{n+1}(x^2-34x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n-2}+32(216x(x^2+68x+1)W_1^3-(42875x^3-1329231x^2+39237x-1)W_0^3-108x(35x^2+1224x+1)W_1^2W_0+18x(1225x^2+2414x+1)W_0^2W_1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^3=\frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2=32x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))W_{2n}^3+32x^n(n(x^2-34x+1)+3x^2-68x+1)W_{2n-2}^3+3x^n(n(x^2-39202x+1)+3x^2-78404x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n+2}-102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)(W_1^2+W_0^2-6W_0W_1)W_{2n}+3x^n(n(x^2-34x+1)+3x^2-68x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n-2}+96(72(3x^2+136x+1)W_1^3-(42875x^2-886154x+13079)W_0^3-36(105x^2+2448x+1)W_1^2W_0+6(3675x^2+4828x+1)W_0^2W_1)$.

(c) $(m=2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3=\frac{\Psi_1}{32\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x-39202)(x^2-34x+1)W_{2n+1}^3+32x^{n+1}(x^2-34x+1)W_{2n-1}^3+3x^{n+1}(x^2-39202x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n+3}-102x^{n+1}\left((x^2-39202x+1153)\right)(W_1^2+W_0^2-6W_0W_1)W_{2n+1}+3x^{n+1}(x^2-34x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n-1}+32((x+1)(x^2+3638x+1)W_1^3-216x(x^2+68x+1)W_0^3-18x(x^2+2414x+1225)W_0W_1^2+108x(x^2+1224x+35)W_0^2W_1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3=\frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 32x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))W_{2n+1}^3+32x^n(n(x^2-34x+1)+3x^2-68x+1)W_{2n-1}^3+3x^n(n(x^2-39\,202x+1)+3x^2-78404x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n+3}-102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)(W_1^2+W_0^2-6W_0W_1)W_{2n+1}+3x^n(n(x^2-34x+1)+3x^2-68x+1)(W_1^2+W_0^2-6W_0W_1)W_{2n-1}+96((x^2+2426x+1213)W_1^3-72(3x^2+136x+1)W_0^3-6(3x^2+4828x+1225)W_1^2W_0+36(3x^2+2448x+35)W_0^2W_1)$.

(d) $(m=-1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_1}{32\,(x^2-6x+1)\,(x^2-198x+1)},$$

where

$\Psi_1 = 32x^{n+1}(x^2-6x+1)W_{-n+1}^3 + 32x^{n+1}(x-198)(x^2-6x+1)W_{-n}^3 + 3x^{n+1}(x^2-6x+1)(W_1^2+W_0^2-6W_0W_1)W_{-n+1} - 18x^{n+1}(x^2-198x+33)(W_1^2+W_0^2-6W_0W_1)W_{-n} + 3x^{n+1}(x^2-198x+1)(W_1^2+W_0^2-6W_0W_1)W_{-n-1} + 32(-x(x^2+12x+1)W_1^3 + (x^2+12x+1)W_0^3 + 18x(6x+1)W_1^2W_0 - 18x(x+6)W_0^2W_1)$,

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_2}{128\,(x^3-153x^2+595x-51)},$$

where

$\Psi_2 = 32x^n(n(x^2-6x+1)+3x^2-12x+1)W_{-n+1}^3 + 32x^n(n(x-198)(x^2-6x+1)+2(2x^3-306x^2+1189x-99))W_{-n}^3 + 3x^n(n(x^2-6x+1)+3x^2-12x+1)(W_1^2+W_0^2-6W_0W_1)W_{-n+1} - 18x^n(n(x^2-198x+33)+3x^2-396x+33)(W_1^2+W_0^2-6W_0W_1)W_{-n} + 3x^n(n(x^2-198x+1)+3x^2-396x+1)(W_1^2+W_0^2-6W_0W_1)W_{-n-1} + 32(-(3x^2+24x+1)W_1^3 + 2(x+6)W_0^3 + 18(12x+1)W_1^2W_0 - 36(x+3)W_0^2W_1)$.

(e) $(m=-2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_1}{32\,(x^2-34x+1)\,(x^2-39202x+1)},$$

where

$\Psi_1 = 32x^{n+1}(x^2-34x+1)W_{-2n+2}^3 + 32x^{n+1}(x-39202)(x^2-34x+1)W_{-2n}^3 + 3x^{n+1}(x^2-34x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n+2} - 102x^{n+1}(x^2-39202x+1153)(W_1^2+W_0^2-6W_0W_1)W_{-2n} + 3x^{n+1}(x^2-39202x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n-2} + 32(-216x(x^2+68x+1)W_1^3 + (x+1)(x^2+3638x+1)W_0^3 + 108x(x^2+1224x+35)W_1^2W_0 - 18x(x^2+2414x+1225)W_0^2W_1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_2}{128\,(x^3-29427x^2+666435x-9809)},$$

where

$\Psi_2 = 32x^n(n(x^2-34x+1)+3x^2-68x+1)W_{-2n+2}^3 + 32x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))W_{-2n}^3 + 3x^n(n(x^2-34x+1)+3x^2-68x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n+2} - 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)(W_1^2+W_0^2-6W_0W_1)W_{-2n} + 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n-2} + 96(-72(3x^2+136x+1)W_1^3 + (x^2+2426x+1213)W_0^3 + 36(3x^2+2448x+35)W_1^2W_0 - 6(3x^2+4828x+1225)W_0^2W_1)$.

(f) $(m=-2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_1}{32\,(x^2-34x+1)\,(x^2-39202x+1)},$$

where

$\Psi_1 = 32x^{n+1}(x^2-34x+1)W^3_{-2n+3} + 32x^{n+1}(x-39202)(x^2-34x+1)W^3_{-2n+1} + 3x^{n+1}(x^2-34x+1)\left(W_0^2+W_1^2-6W_0W_1\right)W_{-2n+3} - 102x^{n+1}(x^2-39202x+1153)(W_1^2+W_0^2-6W_0W_1)W_{-2n+1} + 3x^{n+1}(x^2-39202x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n-1} + 32(-(42875x^3-1329231x^2+39237x-1)W_1^3 + 216x(x^2+68x+1)W_0^3 + 18x(1225x^2+2414x+1)W_0W_1^2 - 108x(35x^2+1224x+1)W_0^2W_1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W^3_{-2k+1} = \frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 32x^n(n(x^2-34x+1)+3x^2-68x+1)W^3_{-2n+3} + 32x^n(n(x-39202)(x^2-34x+1) + 4x^3-117708x^2+2665738x-39202)W^3_{-2n+1} + 3x^n(n(x^2-34x+1)+3x^2-68x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n+3} - 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)(W_1^2+W_0^2-6W_0W_1)W_{-2n+1} + 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)(W_1^2+W_0^2-6W_0W_1)W_{-2n-1} + 96(-(42875x^2-886154x+13079)W_1^3 + 72(3x^2+136x+1)W_0^3 + 6(3675x^2+4828x+1)W_1^2W_0 - 36(105x^2+2448x+1)W_0^2W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of balancing numbers (take $W_n = B_n$ with $B_0 = 0, B_1 = 1$);

Corollary 2. *For* $n \geq 0$, *balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k^3 = \frac{\Psi_1}{32\left(x^2-6x+1\right)\left(x^2-198x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x-198)(x^2-6x+1)B_n^3 + 32x^{n+1}(x^2-6x+1)B^3_{n-1} + 3x^{n+1}(x^2-198x+1)B_{n+1} - 18x^{n+1}(x^2-198x+33)B_n + 3x^{n+1}(x^2-6x+1)B_{n-1} + 32x(x^2+12x+1)$,

and if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k B_k^3 = \frac{\Psi_2}{128\left(x^3-153x^2+595x-51\right)},$$

where

$\Psi_2 = 32x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)B_n^3 + 32x^n(n(x^2-6x+1)+3x^2-12x+1)B^3_{n-1} + 3x^n(n(x^2-198x+1)+3x^2-396x+1)B_{n+1} - 18x^n(n(x^2-198x+33)+3x^2-396x+33)B_n + 3x^n(n(x^2-6x+1)+3x^2-12x+1)B_{n-1} + 32(3x^2+24x+1)$.

(b) $(m=2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k}^3 = \frac{\Psi_1}{32\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x-39202)(x^2-34x+1)B^3_{2n} + 32x^{n+1}(x^2-34x+1)B^3_{2n-2} + 3x^{n+1}(x^2-39202x+1)B_{2n+2} - 102x^{n+1}(x^2-39202x+1153)B_{2n} + 3x^{n+1}(x^2-34x+1)B_{2n-2} + 6912x\left(x^2+68x+1\right)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k B_{2k}^3 = \frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 32x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))B_{2n}^3+32x^n(n(x^2-34x+1)+3x^2-68x+1)B_{2n-2}^3+3x^n(n(x^2-39202x+1)+3x^2-78404x+1)B_{2n+2}-102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)B_{2n}+3x^n(n(x^2-34x+1)+3x^2-68x+1)B_{2n-2}+6912(3x^2+136x+1).$

(c) $(m=2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k+1}^3 = \frac{\Psi_1}{32\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x-39202)(x^2-34x+1)B_{2n+1}^3+32x^{n+1}(x^2-34x+1)B_{2n-1}^3+3x^{n+1}(x^2-39202x+1)B_{2n+3}-102x^{n+1}\left((x^2-39202x+1153)\right)B_{2n+1}+3x^{n+1}(x^2-34x+1)B_{2n-1}+32(x+1)(x^2+3638x+1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k B_{2k+1}^3 = \frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 32x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))B_{2n+1}^3+32x^n(n(x^2-34x+1)+3x^2-68x+1)B_{2n-1}^3+3x^n(n(x^2-39202x+1)+3x^2-78404x+1)B_{2n+3}-102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)B_{2n+1}+3x^n(n(x^2-34x+1)+3x^2-68x+1)B_{2n-1}+96(x^2+2426x+1213).$

(d) $(m=-1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-k}^3 = \frac{\Psi_1}{32\left(x^2-6x+1\right)\left(x^2-198x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x^2-6x+1)B_{-n+1}^3+32x^{n+1}(x-198)(x^2-6x+1)B_{-n}^3+3x^{n+1}(x^2-6x+1)B_{-n+1}-18x^{n+1}(x^2-198x+33)B_{-n}+3x^{n+1}(x^2-198x+1)B_{-n-1}-32x(x^2+12x+1)$,

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k B_{-k}^3 = \frac{\Psi_2}{128\left(x^3-153x^2+595x-51\right)},$$

where

$\Psi_2 = 32x^n(n(x^2-6x+1)+3x^2-12x+1)B_{-n+1}^3+32x^n(n(x-198)(x^2-6x+1)+2(2x^3-306x^2+1189x-99))B_{-n}^3+3x^n(n(x^2-6x+1)+3x^2-12x+1)B_{-n+1}-18x^n(n(x^2-198x+33)+3x^2-396x+33)B_{-n}+3x^n(n(x^2-198x+1)+3x^2-396x+1)B_{-n-1}-32(3x^2+24x+1).$

(e) $(m=-2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k}^3 = \frac{\Psi_1}{32\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x^2-34x+1)B_{-2n+2}^3 + 32x^{n+1}(x-39202)(x^2-34x+1)B_{-2n}^3 + 3x^{n+1}(x^2-34x+1)B_{-2n+2} - 102x^{n+1}(x^2-39202x+1153)B_{-2n} + 3x^{n+1}(x^2-39202x+1)B_{-2n-2} - 6912x(x^2+68x+1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k B_{-2k}^3 = \frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 32x^n(n(x^2-34x+1)+3x^2-68x+1)B_{-2n+2}^3 + 32x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))B_{-2n}^3 + 3x^n(n(x^2-34x+1)+3x^2-68x+1)B_{-2n+2} - 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)B_{-2n} + 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)B_{-2n-2} - 6912(3x^2+136x+1)$.

(f) $(m=-2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k+1}^3 = \frac{\Psi_1}{32\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 32x^{n+1}(x^2-34x+1)B_{-2n+3}^3 + 32x^{n+1}(x-39202)(x^2-34x+1)B_{-2n+1}^3 + 3x^{n+1}(x^2-34x+1)\left(B_0^2+B_1^2-6B_0B_1\right)B_{-2n+3} - 102x^{n+1}(x^2-39202x+1153)B_{-2n+1} + 3x^{n+1}(x^2-39202x+1)B_{-2n-1} - 32(42875x^3-1329231x^2+39237x-1)$,

and if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k B_{-2k+1}^3 = \frac{\Psi_2}{128\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 32x^n(n(x^2-34x+1)+3x^2-68x+1)B_{-2n+3}^3 + 32x^n(n(x-39202)(x^2-34x+1)+4x^3-117708x^2+2665738x-39202)B_{-2n+1}^3 + 3x^n(n(x^2-34x+1)+3x^2-68x+1)B_{-2n+3} - 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)B_{-2n+1} + 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)B_{-2n-1} - 96(42875x^2-886154x+13079)$.

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 6$ in the last proposition, we have the following corollary which presents sum formulas of modified Lucas-balancing numbers;

Corollary 3. *For* $n\geq 0$, *modified Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_k^3 = \frac{\Psi_1}{\left(x^2-6x+1\right)\left(x^2-198x+1\right)},$$

where

$\Psi_1 = x^{n+1}(x-198)(x^2-6x+1)H_n^3 + x^{n+1}(x^2-6x+1)H_{n-1}^3 - 3x^{n+1}(x^2-198x+1)H_{n+1} + 18x^{n+1}(x^2-198x+33)H_n - 3x^{n+1}(x^2-6x+1)H_{n-1} - 8(27x^3-595x^2+177x-1)$,

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_k^3 = \frac{\Psi_2}{4\left(x^3-153x^2+595x-51\right)},$$

where

$\Psi_2 = x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)H_n^3 + x^n(n(x^2-6x+1)+3x^2-12x+1)H_{n-1}^3 - 3x^n(n(x^2-198x+1)+3x^2-396x+1)H_{n+1} + 18x^n(n(x^2-198x+33)+3x^2-396x+33)H_n - 3x^n(n(x^2-6x+1)+3x^2-12x+1)H_{n-1} - 8(81x^2-1190x+177)$.

(b) $(m=2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^3 = \frac{\Psi_1}{\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = x^{n+1}(x-39202)(x^2-34x+1)H_{2n}^3 + x^{n+1}(x^2-34x+1)H_{2n-2}^3 - 3x^{n+1}(x^2-39202x+1)H_{2n+2} + 102x^{n+1}(x^2-39202x+1153)H_{2n} - 3x^{n+1}(x^2-34x+1)H_{2n-2} - 8(4913x^3-666435x^2+34323x-1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_{2k}^3 = \frac{\Psi_2}{4\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))H_{2n}^3 + x^n(n(x^2-34x+1)+3x^2-68x+1)H_{2n-2}^3 - 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)H_{2n+2} + 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)H_{2n} - 3x^n(n(x^2-34x+1)+3x^2-68x+1)H_{2n-2} - 24(4913x^2-444290x+11441)$.

(c) $(m=2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^3 = \frac{\Psi_1}{\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = x^{n+1}(x-39202)(x^2-34x+1)H_{2n+1}^3 + x^{n+1}(x^2-34x+1)H_{2n-1}^3 - 3x^{n+1}(x^2-39202x+1)H_{2n+3} + 102x^{n+1}\left((x^2-39202x+1153)\right)H_{2n+1} - 3x^{n+1}(x^2-34x+1)H_{2n-1} - 216(x-1)(x^2-3298x+1)$,

and if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^3 = \frac{\Psi_2}{4\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))H_{2n+1}^3 + x^n(n(x^2-34x+1)+3x^2-68x+1)H_{2n-1}^3 - 3x^n(n(x^2-39\,202x+1)+3x^2-78404x+1)H_{2n+3} + 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)H_{2n+1} - 3x^n(n(x^2-34x+1)+3x^2-68x+1)H_{2n-1} - 216(3x^2-6598x+3299)$.

(d) $(m=-1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-k}^3 = \frac{\Psi_1}{(x^2-6x+1)(x^2-198x+1)},$$

where

$\Psi_1 = x^{n+1}(x^2-6x+1)H_{-n+1}^3 + x^{n+1}(x-198)(x^2-6x+1)H_{-n}^3 - 3x^{n+1}(x^2-6x+1)H_{-n+1} + 18x^{n+1}(x^2-198x+33)H_{-n} - 3x^{n+1}(x^2-198x+1)H_{-n-1} - 8(27x^3-595x^2+177x-1)$,

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_{-k}^3 = \frac{\Psi_2}{4\,(x^3-153x^2+595x-51)},$$

where

$\Psi_2 = x^n(n(x^2-6x+1)+3x^2-12x+1)H_{-n+1}^3 + x^n(n(x-198)(x^2-6x+1)+2(2x^3-306x^2+1189x-99))H_{-n}^3 - 3x^n(n(x^2-6x+1)+3x^2-12x+1)H_{-n+1} + 18x^n(n(x^2-198x+33)+3x^2-396x+33)H_{-n} - 3x^n(n(x^2-198x+1)+3x^2-396x+1)H_{-n-1} - 8(81x^2-1190x+177)$.

(e) $(m=-2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^3 = \frac{\Psi_1}{(x^2-34x+1)(x^2-39202x+1)},$$

where

$\Psi_1 = x^{n+1}(x^2-34x+1)H_{-2n+2}^3 + x^{n+1}(x-39202)(x^2-34x+1)H_{-2n}^3 - 3x^{n+1}(x^2-34x+1)H_{-2n+2} + 102x^{n+1}(x^2-39202x+1153)H_{-2n} - 3x^{n+1}(x^2-39202x+1)H_{-2n-2} - 8(4913x^3-666435x^2+34323x-1)$,

and if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^3 = \frac{\Psi_2}{4\,(x^3-29427x^2+666435x-9809)},$$

where

$\Psi_2 = x^n(n(x^2-34x+1)+3x^2-68x+1)H_{-2n+2}^3 + x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))H_{-2n}^3 - 3x^n(n(x^2-34x+1)+3x^2-68x+1)H_{-2n+2} + 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)H_{-2n} - 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)H_{-2n-2} - 24(4913x^2-444290x+11441)$.

(f) $(m=-2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^3 = \frac{\Psi_1}{(x^2-34x+1)(x^2-39202x+1)},$$

where

$\Psi_1 = x^{n+1}(x^2-34x+1)H_{-2n+3}^3 + x^{n+1}(x-39202)(x^2-34x+1)H_{-2n+1}^3 - 3x^{n+1}(x^2-34x+1)H_{-2n+3} + 102x^{n+1}(x^2-39202x+1153)H_{-2n+1} - 3x^{n+1}(x^2-39202x+1)H_{-2n-1} - 216(35937x^3-1329571x^2+39235x-1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H^3_{-2k+1} = \frac{\Psi_2}{4\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = x^n(n(x^2-34x+1)+3x^2-68x+1)H^3_{-2n+3} + x^n(n(x-39202)(x^2-34x+1)+4x^3-117708x^2+2665738x-39202)H^3_{-2n+1} - 3x^n(n(x^2-34x+1)+3x^2-68x+1)H_{-2n+3} + 102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)H_{-2n+1} - 3x^n(n(x^2-39202x+1)+3x^2-78404x+1)H_{-2n-1} - 216(107811x^2-2659142x+39235)$.

From the above proposition, we have the following corollary which gives sum formulas of Lucas-balancing numbers (take $W_n = C_n$ with $C_0 = 1, C_1 = 3$);

Corollary 4. *For $n \geq 0$, Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k^3 = \frac{\Psi_1}{4\left(x^2-6x+1\right)\left(x^2-198x+1\right)},$$

where

$\Psi_1 = 4x^{n+1}(x-198)(x^2-6x+1)C_n^3 + 4x^{n+1}(x^2-6x+1)C^3_{n-1} - 3x^{n+1}(x^2-198x+1)C_{n+1} + 18x^{n+1}(x^2-198x+33)C_n - 3x^{n+1}(x^2-6x+1)C_{n-1} - 4(27x^3-595x^2+177x-1)$,

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k C_k^3 = \frac{\Psi_2}{16\left(x^3-153x^2+595x-51\right)},$$

where

$\Psi_2 = 4x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)C_n^3 + 4x^n(n(x^2-6x+1)+3x^2-12x+1)C^3_{n-1} - 3x^n(n(x^2-198x+1)+3x^2-396x+1)C_{n+1} + 18x^n(n(x^2-198x+33)+3x^2-396x+33)C_n - 3x^n(n(x^2-6x+1)+3x^2-12x+1)C_{n-1} - 4(81x^2-1190x+177)$.

(b) $(m=2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.*, $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k}^3 = \frac{\Psi_1}{4\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 4x^{n+1}(x-39202)(x^2-34x+1)C^3_{2n} + 4x^{n+1}(x^2-34x+1)C^3_{2n-2} - 3x^{n+1}(x^2-39202x+1)C_{2n+2} + 102x^{n+1}(x^2-39202x+1153)C_{2n} - 3x^{n+1}(x^2-34x+1)C_{2n-2} - 4(4913x^3-666\,435x^2+34\,323x-1)$,

and

if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.*, $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k C_{2k}^3 = \frac{\Psi_2}{16\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 4x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))C_{2n}^3+4x^n(n(x^2-34x+1)+3x^2-68x+1)C_{2n-2}^3-3x^n(n(x^2-39202x+1)+3x^2-78404x+1)C_{2n+2}+102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)C_{2n}-3x^n(n(x^2-34x+1)+3x^2-68x+1)C_{2n-2}-12(4913x^2-444290x+11441)$.

(c) $(m=2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1}^3 = \frac{\Psi_1}{4\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 4x^{n+1}(x-39202)(x^2-34x+1)C_{2n+1}^3+4x^{n+1}(x^2-34x+1)C_{2n-1}^3-3x^{n+1}(x^2-39202x+1)C_{2n+3}+102x^{n+1}\left((x^2-39202x+1153)\right)C_{2n+1}-3x^{n+1}(x^2-34x+1)C_{2n-1}-108(x-1)(x^2-3298x+1)$,

and if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k C_{2k+1}^3 = \frac{\Psi_2}{16\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 4x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))C_{2n+1}^3+4x^n(n(x^2-34x+1)+3x^2-68x+1)C_{2n-1}^3-3x^n(n(x^2-39202x+1)+3x^2-78404x+1)C_{2n+3}+102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)C_{2n+1}-3x^n(n(x^2-34x+1)+3x^2-68x+1)C_{2n-1}-108(3x^2-6598x+3299)$.

(d) $(m=-1, j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k}^3 = \frac{\Psi_1}{4\left(x^2-6x+1\right)\left(x^2-198x+1\right)},$$

where

$\Psi_1 = 4x^{n+1}(x^2-6x+1)C_{-n+1}^3+4x^{n+1}(x-198)(x^2-6x+1)C_{-n}^3-3x^{n+1}(x^2-6x+1)C_{-n+1}+18x^{n+1}(x^2-198x+33)C_{-n}-3x^{n+1}(x^2-198x+1)C_{-n-1}-4(27x^3-595x^2+177x-1)$,

and if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k C_{-k}^3 = \frac{\Psi_2}{16\left(x^3-153x^2+595x-51\right)},$$

where

$\Psi_2 = 4x^n(n(x^2-6x+1)+3x^2-12x+1)C_{-n+1}^3+4x^n(n(x-198)(x^2-6x+1)+2(2x^3-306x^2+1189x-99))C_{-n}^3-3x^n(n(x^2-6x+1)+3x^2-12x+1)C_{-n+1}+18x^n(n(x^2-198x+33)+3x^2-396x+33)C_{-n}-3x^n(n(x^2-198x+1)+3x^2-396x+1)C_{-n-1}-4(81x^2-1190x+177)$.

(e) $(m=-2, j=0)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k}^3 = \frac{\Psi_1}{4\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 4x^{n+1}(x^2-34x+1)C_{-2n+2}^3+4x^{n+1}(x-39202)(x^2-34x+1)C_{-2n}^3-3x^{n+1}(x^2-34x+1)C_{-2n+2}+102x^{n+1}(x^2-39202x+1153)C_{-2n}-3x^{n+1}(x^2-39202x+1)C_{-2n-2}-4(4913x^3-666435x^2+34323x-1)$, *and if* $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k C^3_{-2k} = \frac{\Psi_2}{16\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 4x^n(n(x^2-34x+1)+3x^2-68x+1)C^3_{-2n+2}+4x^n(n(x-39202)(x^2-34x+1)+2(2x^3-58854x^2+1332869x-19601))C^3_{-2n}-3x^n(n(x^2-34x+1)+3x^2-68x+1)C_{-2n+2}+102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)C_{-2n}-3x^n(n(x^2-39202x+1)+3x^2-78404x+1)C_{-2n-2}-12(4913x^2-444290x+11441)$.

(f) $(m=-2, j=1)$

If $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)\neq 0$, *i.e.,* $x\neq 19601+13860\sqrt{2}$, $x\neq 19601-13860\sqrt{2}$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C^3_{-2k+1} = \frac{\Psi_1}{4\left(x^2-34x+1\right)\left(x^2-39202x+1\right)},$$

where

$\Psi_1 = 4x^{n+1}(x^2-34x+1)C^3_{-2n+3}+4x^{n+1}(x-39202)(x^2-34x+1)C^3_{-2n+1}-3x^{n+1}(x^2-34x+1)C_{-2n+3}+102x^{n+1}(x^2-39202x+1153)C_{-2n+1}-3x^{n+1}(x^2-39202x+1)C_{-2n-1}-108(35937x^3-1329571x^2+39235x-1)$,

and if $\left(x^2-34x+1\right)\left(x^2-39202x+1\right)=0$, *i.e.,* $x=19601+13860\sqrt{2}$ *or* $x=19601-13860\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k C^3_{-2k+1} = \frac{\Psi_2}{16\left(x^3-29427x^2+666435x-9809\right)},$$

where

$\Psi_2 = 4x^n(n(x^2-34x+1)+3x^2-68x+1)C^3_{-2n+3}+4x^n(n(x-39202)(x^2-34x+1)+4x^3-117708x^2+2665738x-39202)C^3_{-2n+1}-3x^n(n(x^2-34x+1)+3x^2-68x+1)C_{-2n+3}+102x^n(n(x^2-39202x+1153)+3x^2-78404x+1153)C_{-2n+1}-3x^n(n(x^2-39202x+1)+3x^2-78404x+1)C_{-2n-1}-108(107811x^2-2659142x+39235)$.

Taking $x=1$ in the last two corollaries we get the following corollary;

Corollary 5. *For* $n\geq 0$, *balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers have the following properties:*

1.

(a) $\sum\limits_{k=0}^{n} B_k^3 = \frac{1}{6272}(6304B_n^3-32B_{n-1}^3-147B_{n+1}+738B_n-3B_{n-1}+112)$.

(b) $\sum\limits_{k=0}^{n} B_{2k}^3 = \frac{1}{1254400}(1254432B_{2n}^3-32B_{2n-2}^3-3675B_{2n+2}+121278B_{2n}-3B_{2n-2}+15120)$.

(c) $\sum\limits_{k=0}^{n} B_{2k+1}^3 = \frac{1}{1254400}(1254432B_{2n+1}^3-32B_{2n-1}^3-3675B_{2n+3}+121278B_{2n+1}-3B_{2n-1}+7280)$.

(d) $\sum\limits_{k=0}^{n} B_{-k}^3 = \frac{1}{6272}(6304B_{-n}^3-32B_{-n+1}^3-3B_{-n+1}+738B_{-n}-147B_{-n-1}-112)$.

(e) $\sum\limits_{k=0}^{n} B_{-2k}^3 = \frac{1}{1254400}(-32B_{-2n+2}^3+1254432B_{-2n}^3-3B_{-2n+2}+121278B_{-2n}-3675B_{-2n-2}-15120)$.

(f) $\sum\limits_{k=0}^{n} B_{-2k+1}^3 = \frac{1}{1254400}(-32B_{-2n+3}^3+1254432B_{-2n+1}^3-3B_{-2n+3}+121278B_{-2n+1}-3675B_{-2n-1}+1247120)$.

2.

(a) $\sum\limits_{k=0}^{n} H_k^3 = \frac{1}{196}(197H_n^3-H_{n-1}^3+147H_{n+1}-738H_n+3H_{n-1}+784)$.

(b) $\sum_{k=0}^{n} H_{2k}^3 = \frac{1}{39200}(39201H_{2n}^3 - H_{2n-2}^3 + 3675H_{2n+2} - 121278H_{2n} + 3H_{2n-2} + 156800)$.

(c) $\sum_{k=0}^{n} H_{2k+1}^3 = \frac{1}{39200}(39201H_{2n+1}^3 - H_{2n-1}^3 + 3675H_{2n+3} - 121278H_{2n+1} + 3H_{2n-1})$.

(d) $\sum_{k=0}^{n} H_{-k}^3 = \frac{1}{196}(197H_{-n}^3 - H_{-n+1}^3 + 3H_{-n+1} - 738H_{-n} + 147H_{-n-1} + 784)$.

(e) $\sum_{k=0}^{n} H_{-2k}^3 = \frac{1}{39200}(-H_{-2n+2}^3 + 39201H_{-2n}^3 + 3H_{-2n+2} - 121278H_{-2n} + 3675H_{-2n-2} + 156800)$.

(f) $\sum_{k=0}^{n} H_{-2k+1}^3 = \frac{1}{39200}(-H_{-2n+3}^3 + 39201H_{-2n+1}^3 + 3H_{-2n+3} - 121278H_{-2n+1} + 3675H_{-2n-1} + 8467200)$.

3.

(a) $\sum_{k=0}^{n} C_k^3 = \frac{1}{784}(788C_n^3 - 4C_{n-1}^3 + 147C_{n+1} - 738C_n + 3C_{n-1} + 392)$.

(b) $\sum_{k=0}^{n} C_{2k}^3 = \frac{1}{156800}(156804C_{2n}^3 - 4C_{2n-2}^3 + 3675C_{2n+2} - 121278C_{2n} + 3C_{2n-2} + 78400)$.

(c) $\sum_{k=0}^{n} C_{2k+1}^3 = \frac{1}{156800}(156804C_{2n+1}^3 - 4C_{2n-1}^3 + 3675C_{2n+3} - 121278C_{2n+1} + 3C_{2n-1})$.

(d) $\sum_{k=0}^{n} C_{-k}^3 = \frac{1}{784}(-4C_{-n+1}^3 + 788C_{-n}^3 + 3C_{-n+1} - 738C_{-n} + 147C_{-n-1} + 392)$.

(e) $\sum_{k=0}^{n} C_{-2k}^3 = \frac{1}{156800}(-4C_{-2n+2}^3 + 156804C_{-2n}^3 + 3C_{-2n+2} - 121278C_{-2n} + 3675C_{-2n-2} + 78400)$.

(f) $\sum_{k=0}^{n} C_{-2k+1}^3 = \frac{1}{156800}(-4C_{-2n+3}^3 + 156804C_{-2n+1}^3 + 3C_{-2n+3} - 121278C_{-2n+1} + 3675C_{-2n-1} + 4233600)$.

3. Conclusions

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized balancing sequence, and then we have presented the formulas as special cases the corresponding identity for the balancing, modified Lucas-balancing and Lucas-balancing numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

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