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Pulse Vaccination Strategy in an Epidemic Model with Two Susceptible Subclasses and Time Delay

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Abstract

In this paper, an impulsive epidemic model with time delay is proposed, which susceptible population is divided into two groups: high risk susceptibles and non-high risk susceptibles. We introduce two thresholds R_1, R_2 and demonstrate that the disease will be extinct if $R_1 < 1$ and persistent if $R_2 > 1$. Our results show that larger pulse vaccination rates or a shorter the period of pulsing will lead to the eradication of the disease. The conclusions are confirmed by numerical simulations.

Keywords: High Risk Susceptible, Non-High Risk Susceptible, Pulse Vaccination, Extinction

1. Introduction

Infectious diseases have tremendous influence on human life. Every year, millions of people die of various infectious diseases. Controlling infectious diseases has been an increasingly complex issue in recent years [1]. Over the last fifty years, many scholars have payed great attention to construct mathematical models to describe the spread of infectious diseases. See the literatures [2-8], the books [9-11] and the references therein. In the classical epidemiological models, a population of total size N is divided into S (susceptible numbers), I (infective numbers), or S, I and R (recovered numbers) or S, E (exposed numbers), I and R, and corresponding epidemiological models such as *SI*, *SIS*, *SIR*, *SIRS*, *SIER* and *SEIRS* are constructed. All these models are extensions of the *SIR* model elaborated by Kermack and McKendrick in 1927 [12]. Anderson and May [5,9] discussed the spreading nature of biological viruses, parasites etc. leading to infectious diseases in human population through several epidemic models. Cooke and Driessche [8] investigated an *SEIRS* model with the latent period and the immune period. The consideration of the latent period and the immune period gave rise to models with the incorporation of delays and integral equation formulations.

However, owing to the physical health status, age and other factors, susceptibles population show different infective to a infectious disease. In this paper, we divide the susceptible population into two groups: nonhigh risk susceptibles (S_1) and high risk susceptibles (S_2), such that individuals in each group have homogeneous susceptibility, but the susceptibilities of individuals from different groups are distinct. In this paper, we propose a new SIR epidemic model, which two noninteracting susceptible subclasses, the nonlinear incidence $\beta S^p I$, time delay and pulse vaccination are considered. The main purpose of this paper is to study the dynamical behavior of the model and establish sufficient conditions that the disease will be extinct or not.

The organization of this paper is as follows. In the next section, we construct a delayed and impulsive *SIR* epidemic model with two noninteracting susceptible subclasses. In Section 3, using the discrete dynamical system determined by the stroboscopic map, we establish sufficient conditions for the global attractivity of infection-free periodic solution. And the sufficient conditions for the permanence of the model are obtained in section 4. Finally, we present some numerical simulations to illustrate our results.

2. Model Formulation and Preliminaries

Gao etc. [13] proposed a delayed *SIR* epidemic model with pulse vaccination:

$$\begin{cases} \left.\begin{aligned} S'(t) &= \mu - \beta S(t)I(t) - \mu S(t), \\ I'(t) &= \beta S(t)I(t) - \beta e^{-\mu\tau} S(t-\tau)I(t-\tau) - \mu I(t), \\ R'(t) &= \beta e^{-\mu\tau} S(t-\tau)I(t-\tau) - \mu R(t), \end{aligned}\right\} t \neq nT, n \in N \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ I(t^+) &= I(t), \\ R(t^+) &= R(t) + \theta S(t), \end{aligned}\right\} \quad t = nT, n \in N \end{cases} \tag{1}$$

In model (1), the authors assumed that the birth rate (μ) is equal to the death rate, and use a bilinear incidence rate. Motivated by [13], in this paper, we assume that there are two cases noninteracting susceptible subclasses. We denote the density of the susceptible individuals that belong to different subclasses, the infected individuals, and the recovered individuals in the population by S_1, S_2, I and R, respectively, that is, the total variable population $N = S_1 + S_2 + I + R$. Moreover, if the nonlinear incidence $\beta IS^p (p \geq 1)$, different constant recruitments and death rates are incorporated into model (1). Then the corresponding model is investigated:

$$\begin{cases} \left.\begin{aligned} S_1'(t) &= \lambda_1 - \beta_1 S_1^p(t) I(t) - \mu_1 S_1(t), \\ S_2'(t) &= \lambda_2 - \beta_2 S_2^p(t) I(t) - \mu_2 S_2(t), \\ I'(t) &= \beta_1 S_1^p(t) I(t) + \beta_2 S_2^p(t) I(t) - \beta_1 e^{-d\tau} S_1^p(t-\tau) I(t-\tau) - \beta_2 e^{-d\tau} S_2^p(t-\tau) I(t-\tau) - dI(t), \\ R'(t) &= \beta_1 e^{-d\tau} S_1^p(t-\tau) I(t-\tau) + \beta_2 e^{-d\tau} S_2^p(t-\tau) I(t-\tau) - \sigma R(t), \end{aligned}\right\} t \neq nT, n \in N \\ \left.\begin{aligned} S_1(t^+) &= (1-\theta_1) S_1(t), \\ S_2(t^+) &= (1-\theta_2) S_2(t), \\ I(t^+) &= I(t), \\ R(t^+) &= R(t) + \theta_1 S_1(t) + \theta_2 S_2(t), \end{aligned}\right\} \quad t = nT, n \in N \end{cases} \tag{2}$$

where λ_i, $(i = 1,2)$ are the recruitment rate into the *i*-th susceptible class, respectively parameters μ_i, $(i = 1,2)$, d and σ are the death rates of the susceptible, infected and recovered individuals, and β_i, $(i = 1,2)$ are the contact rates, τ is the infectious period. The term $\beta_i e^{-d\tau} S_i^p(t-\tau) I(t-\tau)$, $(i = 1,2)$ reflects the fact that an individual has recovered from the infective compartments and are still alive after infectious period τ. θ_i, $(i = 1,2)$ are the vaccination rates, and T is the period of pulsing.

Adding all the equations in model (2), the total variable population size is given by the differential equation

$$N'(t) = \lambda_1 + \lambda_2 - \mu_1 S_1(t) - \mu_2 S_2(t) - dI(t) - \sigma R(t).$$

and we have

$$N'(t) \leq \lambda_1 + \lambda_2 - hN(t).$$

where $h = \min\{\mu_1, \mu_2, d, \sigma\}$. It follows that

$$\limsup_{t\to\infty} N(t) \leq \frac{\lambda_1 + \lambda_2}{h}.$$

Note that the first three equations of system (2) do not depend on the forth equation. Thus, we restrict our attention to the following reduced system:

$$\begin{cases} \left.\begin{aligned} S_1'(t) &= \lambda_1 - \beta_1 S_1^p(t) I(t) - \mu_1 S_1(t), \\ S_2'(t) &= \lambda_2 - \beta_2 S_2^p(t) I(t) - \mu_2 S_2(t), \\ I'(t) &= \beta_1 S_1^p(t) I(t) + \beta_2 S_2^p(t) I(t) - \beta_1 e^{-d\tau} S_1^p(t-\tau) I(t-\tau) - \beta_2 e^{-d\tau} S_2^p(t-\tau) I(t-\tau) - dI(t), \end{aligned}\right\} t \neq nT, n \in N \\ \left.\begin{aligned} S_1(t^+) &= (1-\theta_1) S_1(t), \\ S_2(t^+) &= (1-\theta_2) S_2(t), \\ I(t^+) &= I(t), \end{aligned}\right\} \quad t = nT, n \in N \end{cases} \tag{3}$$

The initial conditions of (3) are

$$S_1(t) = \phi_1(t), S_2(t) = \phi_2(t), I(t) = \phi_3(t), \text{ for } t \in [-\tau, 0], \tag{4}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in PC_+$, and PC_+ is the space of all piecewise functions $\phi: [-\tau, 0] \to R_+^3$ with points of discontinuity at $-nT (n \in N)$ of the first kind and

which are continuous from the left, *i.e.*, $\phi(-nT-0)=\phi(-nT)$, and $R_+^3=\{(x_1,x_2,x_3)\in R^3 \mid x_i\geq 0,\ i=1,2,3\}$.

Define a subset of R_+^3

$$\Omega=\left\{\left(S_1(t),S_2(t),I(t)\right)\in R_+^3 \;\middle|\; 0\leq S_1(t)+S_2(t)+I(t)\leq \frac{\lambda_1+\lambda_2}{h}\right\}$$

From biological considerations, we discuss system (3) in the closed set. It is easy to verify that Ω is positively invariant with respect to system (3).

3. Global Attractivity of Disease-Free Periodic Solution

To prove our main results, we state some notations and lemmas which will be essential to our proofs.

Lemma 1 (see [6]) *Consider the following impulsive differential equation*

$$\begin{cases} u'(t)=a-bu(t), t\neq nT,\\ u(t^+)=(1-\theta)u(t),\ t=nT,\end{cases}$$

where $a>0,b>0,0<\theta<1$. *Then above system exists a unique positive periodic solution given by*

$$u^*(t)=\frac{a}{b}+\left(\overline{u}-\frac{a}{b}\right)e^{-b(t-nT)},\ \text{for}\ nT<t\leq(n+1)T,$$

which is globally asymptotically stable, where

$$\overline{u}=\frac{a}{b}\frac{(1-\theta)\left(1-e^{-bT}\right)}{1-(1-\theta)e^{-bT}}.$$

Definition 1 (see [14]). *Let* $V:R_+\times R_+^3\to R_+$, *then V is said to belong to class* V_0 *if*

i) *V is continuous in* $\left(nT,(n+1)T\right]\times R_+^3$ *and for each* $X\in R_+^3$, $\lim\limits_{(t,y)\to(nT^+,X)}V(t,y)=V(nT^+,X)$ *exists.*

ii) *V is locally Lipschitzian in X.*

Lemma 2 (*see* [14]). *Let* $V\in V_0$. *Assume that*

$$\begin{cases} D^+V(t,x)\leq g(t,V(t,x)), & t\neq nT,\\ V(t,x(t^+))\leq \varphi_n(V(t,x)), & t=nT,\end{cases}$$

where $g:R_+\times R_+\to R$ *is continuous in* $\left(nT,(n+1)T\right]\times R_+$ *and for* $u\in R_+$, $n\in N$, $\lim\limits_{(t,y)\to(nT^+,u)}V(t,y)=V(nT^+,u)$ *exists,* $\varphi_n:R_+\to R_+$ *is non-decreasing. Let* $r(t)$ *be the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} u'(t)=g(t,u(t)), & t\neq nT,\\ u(t^+)=\varphi_n(u(t)), & t=nT,\\ u(0^+)=u_0,\end{cases}$$

existing on $[0,\infty]$. *Then* $V(0^+,x_0)\leq u_0$ *implies that* $V(t,x(t))\leq r(t), t\geq 0$, *where* $x(t)$ *is any solution of* (3).

In the following we shall demonstrate that the disease-free periodic solution $(S_1^*(t),S_2^*(t),0)$ is global attractive. We firstly show the existence of the disease-free periodic solution, in which the infectious individuals are entirely absent from the population permanently, *i.e.* $I(t)\equiv 0$ for all $t>0$. Under this condition, the growth of the *i*-th $(i=1,2)$ susceptible individuals must satisfy

$$\begin{cases} S_i'(t)=\lambda_i-\mu_iS_i(t), & t\neq nT,\\ S_i(t^+)=(1-\theta)S_i(t), & t=nT.\end{cases}$$

According to Lemma 1, we know that the periodic solution of the system

$$S_i^*(t)=\frac{\lambda_i}{\mu_i}\left(1-\frac{\theta_ie^{-\mu_i(t-nT)}}{1-(1-\theta_i)e^{-\mu_iT}}\right)$$
$$\text{for}\quad nT<t\leq(n+1)T,\quad (i=1,2)$$

is globally asymptotically stable. Therefore system (3) has a unique disease-free periodic solution

$$(S_1^*(t),S_2^*(t),0).$$

Denote

$$R_1=\frac{\beta_1\left(\frac{\lambda_1}{\mu_1}\frac{1-e^{-\mu_1T}}{1-(1-\theta_i)e^{-\mu_1T}}\right)^p+\beta_2\left(\frac{\lambda_2}{\mu_2}\frac{1-e^{-\mu_2T}}{1-(1-\theta_2)e^{-\mu_2T}}\right)^p}{d}. \tag{5}$$

Theorem 1 If $R_1<1$, then the disease-free periodic solution $(S_1^*(t),S_2^*(t),0)$ of system (3) is globally attractive.

Proof. Since $R_1<1$, we can choose $\varepsilon>0$ sufficiently small such that

$$\beta_1\left(\frac{\lambda_1}{\mu_1}\frac{1-e^{-\mu_1T}}{1-(1-\theta_i)e^{-\mu_1T}}+\varepsilon\right)^p+\beta_2\left(\frac{\lambda_2}{\mu_2}\frac{1-e^{-\mu_2T}}{1-(1-\theta_2)e^{-\mu_2T}}+\varepsilon\right)^p<d. \tag{6}$$

From the first equation and the second equation of system (3), we have $S_i'(t)<\lambda_i-\mu_iS_i(t)$, $(i=1,2)$. Then we consider the following impulsive comparison system

$$\begin{cases} u_i'(t) = \lambda_i - \mu_i u_i(t), t \neq nT, \\ u_i(t^+) = (1-\theta) u_i(t), \ t = nT, \end{cases} \tag{7}$$

According to Lemma 1, we obtain the periodic solution of system (7)

$$u_i^*(t) = \frac{\lambda_i}{\mu_i}\left(1 - \frac{\theta_i e^{-\mu_i(t-nT)}}{1-(1-\theta_i)e^{-\mu_i T}}\right),$$
$$\text{for} \quad nT < t \le (n+1)T, \quad (i=1,2),$$

which is globally asymptotically stable. By the comparison theorem [14], we have that there exists $n_1 \in Z_+$ such that for $nT < t \le (n+1)T$, $n > n_1$

$$S_i(t) < u_i^*(t) + \varepsilon \le \frac{\lambda_i}{\mu_i}\left(1 - \frac{\theta_i e^{-\mu_i T}}{1-(1-\theta_i)e^{-\mu_i T}}\right) + \varepsilon = \bar{S}_i, \quad (i=1,2). \tag{8}$$

Furthermore, from the third equation of system (3), we get $I'(t) < \left(\beta_1 \bar{S}_1^p + \beta_2 \bar{S}_2^p - d\right) I(t)$ for $t > nT$ and $n > n_1$.

From (6) and (8), we have $I'(t) < 0$, then $\lim_{t\to\infty} I(t) = 0$, *i.e.*, for any sufficiently small $\varepsilon_1 > 0$, there exists an integer $n_2 > n_1$ such that $I(t) < \varepsilon_1$, for all $t > n_2 T$.

From the first equation and the second equation of system (3), we have for $t > n_2 T$

$$S_i'(t) > \lambda_i - \left(\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i\right) S_i(t), \quad (i=1,2).$$

Then we consider the following impulsive comparison systems

$$\begin{cases} v_i'(t) = \lambda_i - \left(\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i\right) v_i(t), t \neq nT, \\ v_i(t^+) = (1-\theta_i) v_i(t), t = nT. \end{cases} \tag{9}$$

From Lemma 1, we obtain the periodic solution of system (9)

$$v_i^*(t) = \frac{\lambda_i}{\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i}\left(1 - \frac{\theta_i e^{-\left(\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i\right)(t-nT)}}{1-(1-\theta_i)e^{-\left(\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i\right)T}}\right), \quad nT < t \le (n+1)T, \quad (i=1,2),$$

which is globally asymptotically stable. In view of the comparison theorem [14], there exists an integer $n_3 > n_2$ such that for $nT < t \le (n+1)T$, $n > n_3$,

$$S_i(t) > v_i^*(t) + \varepsilon \ge \frac{\lambda_i}{\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i}\left(1 - \frac{\theta_i e^{-\left(\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i\right)T}}{1-(1-\theta_i)e^{-\left(\beta_i \bar{S}_i^{p-1}\varepsilon_1 + \mu_i\right)T}}\right) + \varepsilon, (i=1,2) \tag{10}$$

Since ε and ε_1 are sufficiently small, from (8) and (10), we know that

$$\lim_{t\to\infty} S_i(t) = S^*(t), \quad (i=1,2).$$

Hence, disease-free periodic solution $\left(S_1^*(t), S_2^*(t), 0\right)$ of system (3) is globally attractive. The proof is completed.

Next, we give some accounts of the Theorem 1 for a well biological meaning.

By simple calculation, from (5) we get

$$\frac{\partial R_1}{\partial \theta_1} = -p\frac{\beta_1}{d}\frac{\lambda_1^p}{\mu_1^p}\frac{\left(1-e^{-\mu_1 T}\right)^p}{\left(1-(1-\theta_1)e^{-\mu_1 T}\right)^{p+1}}e^{-\mu_1 T} < 0,$$
$$\frac{\partial R_1}{\partial \theta_2} = -p\frac{\beta_2}{d}\frac{\lambda_2^p}{\mu_2^p}\frac{\left(1-e^{-\mu_2 T}\right)^p}{\left(1-(1-\theta_2)e^{-\mu_2 T}\right)^{p+1}}e^{-\mu_2 T} < 0, \tag{11}$$

and

$$\frac{\partial R_1}{\partial T} = p\frac{\beta_1}{d}\frac{\lambda_1^p}{\mu_1^p}\frac{\left(1-e^{-\mu_1 T}\right)^{p-1}}{\left(1-(1-\theta_1)e^{-\mu_1 T}\right)^{p+1}}\theta_1\mu_1 e^{-\mu_1 T} + p\frac{\beta_2}{d}\frac{\lambda_2^p}{\mu_2^p}\frac{\left(1-e^{-\mu_2 T}\right)^{p-1}}{\left(1-(1-\theta_2)e^{-\mu_2 T}\right)^{p+1}}\theta_2\mu_2 e^{-\mu_2 T} > 0. \tag{12}$$

Theorem 1 determines the global attravtivity of the disease-free periodic solution of system (3) in Ω for the case $R_1 < 1$. Its epidemiology implies that the disease will die out. From (11) and (12), we can see that larger pulse vaccination rates or a shorter period of immune vaccination will make for the disease eradication.

4. Permanence

In this section, we state the disease is endemic if the infectious population persists above a certain positive level for sufficiently large time. The endemicity of the disease can be well captured and studied through the notation of uniform persistence.

Definition 2. System (3) is said to be uniformly persistent if there exist positive constants $M_i \ge m_i > 0$ $(i=1,2,3)$ (both are independent of the initial values), such that every solution $\left(S_1(t), S_2(t), I(t)\right)$ with positive initial conditions of system (3) satisfies

$$m_1 \le S_1(t) \le M_1, \ m_2 \le S_2(t) \le M_2, \ m_3 \le I(t) \le M_3.$$

Denote

$$R_2=\frac{\beta_1\left(\frac{\lambda_1}{\mu_1}\frac{1-e^{-\mu_1 T}}{1-(1-\theta_1)e^{-\mu_1 T}}\right)^p\left(1-e^{-d\tau}\right)+\beta_2\left(\frac{\lambda_2}{\mu_2}\frac{1-e^{-\mu_2 T}}{1-(1-\theta_2)e^{-\mu_2 T}}\right)^p\left(1-e^{-d\tau}\right)}{d}.$$

Theorem 2. If $R_2>1$, then system (3) is uniformly persistent.

Proof. Let $\left(S_1(t),S_2(t),I(t)\right)$ be any solution of (3) with initial conditions (4), then it is easy to see that

$$S_1(t)\le\frac{\lambda_1+\lambda_2}{h},\quad S_2(t)\le\frac{\lambda_1+\lambda_2}{h},\quad I(t)\le\frac{\lambda_1+\lambda_2}{h},\quad \text{for all}\quad t>0.$$

We are left to prove there exist positive constants m_1,m_2,m_3 such that $S_1(t)\ge m_1, S_2(t)\ge m_2, I(t)\ge m_3$, for all sufficiently large *t*.

Firstly, from the first and second equations of system (3), we have

$$S_i'(t)\ge\lambda_i-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p+\mu_i\right)S_i(t),\ (i=1,2).$$

Considering the following comparison equations

$$\begin{cases}u_i'(t)=\lambda_i-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p+\mu_i\right)u_i(t), t\ne nT,\\ u_i(t^+)=(1-\theta_i)u_i(t),\ t=nT,\end{cases}$$

According to Lemma 1 and the comparison theorem, we know that for any sufficiently small $\tilde{\varepsilon}>0$, there exists a t_0 such that for $t>t_0$,

$$S_i(t)\ge u_i(t)\ge u_i^*(t)-\tilde{\varepsilon}\ge\frac{\lambda_i}{\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p+\mu_i}\left(1-\frac{\theta_i e^{-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p+\mu_i\right)T}}{1-(1-\theta)_i\, e^{-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p+\mu_i\right)T}}\right)-\tilde{\varepsilon}=m_i.$$

Now, we shall prove there exist a $m_3>0$ such that $I(t)\ge m_3$ for all sufficiently large *t*. For convenience, we prove it through the following two steps:

Step I. Since $R_2>1$, there exist sufficiently small $m_I^*>0$ and $\bar{\varepsilon}>0$ such that

$$\beta_1\eta_1^p\left(1-e^{-d\tau}\right)+\beta_2\eta_2^p\left(1-e^{-d\tau}\right)>d, \tag{13}$$

where

$$\eta_i=\frac{\lambda_i}{\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p m_I^*+\mu_i}\left(1-\frac{\theta_i e^{-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p m_I^*+\mu_i\right)T}}{1-(1-\theta_i)e^{-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^p m_I^*+\mu_i\right)T}}\right)-\bar{\varepsilon},\quad (i=1,2).$$

For m_I^*, we can claim that there exists a $t_1>0$ such that $I(t_1)\ge m_I^*$. Otherwise, $I(t)<m_I^*$ for all $t>0$. It follows from the first and second equations of (3) that

$$S_i'(t)\ge\lambda_i-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^{p-1}m_I^*+\mu_i\right)S_i(t),\ (i=1,2).$$

Considering the following impulsive comparison systems

$$\begin{cases}v_i'(t)=\lambda_i-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^{p-1}m_I^*+\mu_i\right)v_i(t), t\ne nT,\\ v_i(t^+)=(1-\theta_i)v_i(t),\ t=nT,\end{cases}$$

Similarly, we know that there exists $t_2>0$, such that for $t>t_2$

$$S_i(t)\ge v_i(t)\ge v_i^*(t)-\tilde{\varepsilon}\ge\frac{\lambda_i}{\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^{p-1}m_I^*+\mu_i}\left(1-\frac{\theta_i e^{-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^{p-1}m_I^*+\mu_i\right)T}}{1-(1-\theta_i)e^{-\left(\beta_i\left(\frac{\lambda_1+\lambda_2}{h}\right)^{p-1}m_I^*+\mu_i\right)T}}\right)-\bar{\varepsilon}=\eta_i. \tag{14}$$

Further, the third equations of system (3) can be rewritten as

$$\begin{aligned}I'(t)&=\beta_1 S_1^p(t)I(t)+\beta_2 S_2^p(t)I(t)\\&\quad-\beta_1 e^{-d\tau}S_1^p(t-\tau)I(t-\tau)\\&\quad-\beta_2 e^{-d\tau}S_2^p(t-\tau)I(t-\tau)-dI(t)\\&=\beta_1 S_1^p(t)I(t)\left(1-e^{-d\tau}\right)+\beta_2 S_2^p(t)I(t)\left(1-e^{-d\tau}\right)\\&\quad-dI(t)+\beta_1 e^{-d\tau}\frac{d}{dt}\int_{t-\tau}^{t}S_1^p(\xi)I(\xi)d\xi\end{aligned}$$

$$+\beta_2 e^{-d\tau}\frac{d}{dt}\int_{t-\tau}^{t} S_2^{p}(\xi)I(\xi)d\xi$$

Define

$$V(t)=I(t)-\beta_1 e^{-d\tau}\int_{t-\tau}^{t} S_1^{p}(\xi)I(\xi)d\xi$$

$$-\beta_2 e^{-d\tau}\int_{t-\tau}^{t} S_2^{p}(\xi)I(\xi)d\xi.$$

For $t>t_2$, the derivative of $V(t)$ along the solution of system (3) is

$$V'(t)=\left[\beta_1 S_1^{p}(t)\left(1-e^{-d\tau}\right)+\beta_2 S_2^{p}(t)\left(1-e^{-d\tau}\right)-d\right]I(t)$$

$$>\left[\beta_1\eta_1^{p}\left(1-e^{-d\tau}\right)+\beta_2\eta_2^{p}\left(1-e^{-d\tau}\right)-d\right]I(t)$$

From (13), we have $V'(t)>0$ for $t>t_2$, which implies that $V(t)\to\infty$ as $t\to\infty$. This contradicts the boundedness of $V(t)$. Hence, there exists a $t_1>0$ such that $I(t_1)\ge m_I^{*}$.

Step II. According to step I, for any positive solution $(S_1(t),\ S_2(t),\ I(t))$ of (3), we are left to consider two cases. First, If $I(t)>m_I^{*}$ for all $t>t_1$, then our result is proved. Second, if $I(t)<m_I^{*}$ for some $t>t_1$,we can choose constants $\rho>0$ and $T_0>max\{t_2,t_1+\tau\}$ (T_0 is sufficiently large) such that $I(t)\le m_I^{*}$, $I(T_0)=m_I^{*}$, $I(T_0+\rho)=m_I^{*}$ and $S_1(t)>\eta_1,\ S_2(t)>\eta_2$ for $t\in[T_0,T_0+\rho]$.

Thus, there exists a $g(0<g<\tau)$ such that for $t\in[T_0,T_0+g]$

$$I(t)\ge\frac{m_I^{*}}{2}. \tag{15}$$

In this case, we shall discuss three possible cases in term of the sizes of g,ρ and τ.

Case 1. If $\rho\le g<\tau$, then it is obvious that $I(t)\ge m_I^{*}/2$, for $t\in[T_0,T_0+\rho]$.

Case 2. If $g<\rho\le\tau$, then from the third equation of (3), we can deduce

$$I(t)=\int_{t-\tau}^{t}\left(\beta_1 S_1^{p}(\xi)+\beta_2 S_2^{p}(\xi)\right)I(\xi)e^{-d(t-\xi)}d\xi, \tag{16}$$

From (14), (15) and (16), we have

$$I(t)>\int_{T_0}^{T_0+g}\left(\beta_1 S_1^{p}(\xi)+\beta_2 S_2^{p}(\xi)\right)I(\xi)e^{-d(t-\xi)}d\xi,\ \text{for } t\in[T_0+g,T_0+\rho]$$

$$>\left(\beta_1\eta_1^{p}+\beta_2\eta_2^{p}\right)\frac{m_I^{*}}{2}e^{-d\tau}g=m_I^{**}$$

Set $m_3=\min\left\{\frac{m_I^{*}}{2},m_I^{**}\right\}>0$, for $t\in[T_0,T_0+\rho]$, we have $I(t)>m_3$.

Case 3. If $g<\tau\le\rho$, we will discuss the following two cases, respectively.

Case 3.1. For $t\in[T_0,T_0+\tau]$, it is easy to obtain $I(t)>m_I^{**}$.

Case 3.2. For $t\in[T_0+\tau,T_0+\rho]$. We claim that $I(t)>m_I^{**}$. Otherwise, there exists a $\bar{t}\in[T_0+\tau,T_0+\rho]$, such that $I(t)>m_I^{**}$ for $t\in[T_0+\tau,\bar{t}]$, and $I(\bar{t})=m_I^{**}$. From (13), (14) and (16), we have

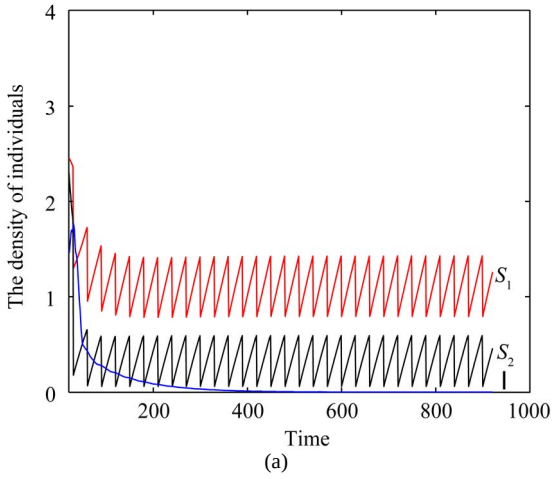
$$I(\bar{t})=\int_{\bar{t}-\tau}^{\bar{t}}\left(\beta_1 S_1^{p}(\xi)+\beta_2 S_2^{p}(\xi)\right)I(\xi)e^{-d(\bar{t}-\xi)}d\xi>\left(\beta_1\eta_1^{p}+\beta_2\eta_2^{p}\right)m_I^{**}\frac{1-e^{-d\tau}}{d}>m_I^{**},$$

which is contradictory to $I(\bar{t})=m_I^{**}$. Hence, the claim holds true.

According to the arbitrary of T_0, we can obtain that $I(t)>m_3$ holds for all $t>T_0$. The proof is completed.

5. Numerical Simulations

In this section, we give some numerical simulations to illustrate the effects of different probability on population. In system (3), $\lambda_1=0.25$, $\lambda_2=0.2$, $\beta_1=0.03$, $\beta_2=0.1$, $\mu_1=0.03$, $\mu_2=0.05$, $p=2$, $d=0.1$, $\tau=2$, $T=3$. Time series are drawn in **Figure 1(a)** and **Figure 1(b)** with initial values $\phi_1(t)=2+0.5\sin t$, $\phi_2(t)=2+0.8\cos t$, $\phi_3(t)=1+0.5\sin t$, $t\in[-2,0]$ for 30 pulsing cycles. If we take $\theta_1=0.45$, $\theta_2=0.90$, then $R_1=0.9956$. By Theorem 1, we know that the disease will disappear (see **Figure 1(a)**). If we let $\theta_1=0.10$, $\theta_2=0.20$, then $R_2=1.4685$. According to Theorem 2, we know that the disease will be permanent (see **Figure 1(b)**).



(a)

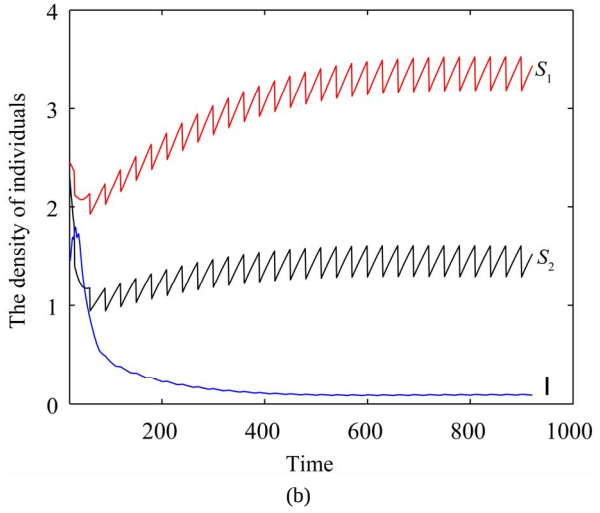


Figure 1. Two figures show that movement paths of S_1, S_2 and I as functions of time t. (a) Disease will be extinct with $R_1 = 0.9956$ and $\theta_1 = 0.45$, $\theta_2 = 0.9$; (b) Disease will be persistent with $R_2 = 1.4685$ and $\theta_1 = 0.1$, $\theta_2 = 0.2$.

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