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On Some Integral Inequalities for Twice Diferentiable $\varphi-$ Convex and Quasi$-$ Convex Functions via $k-$ Fractional Integrals

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, by using the integral equality which is given in this work, we aim at establishing some new inequalities of the Simpson-like and the Hadamard-like type for functions whose second derivatives are φ-convex and quasi-convex.

Keywords: $h-$convex functions; Quasi- convex function; Hermite-Hadamard type inequalities; Simpson's inequality; k-gamma functions; k-beta function.

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1 Introduction

The following inequality is called Hermite-Hadamard inequlity

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$.

The following inequality also is called as Hermite-Hadamard inequlity for fractional integrals.

Let $f : [a, b] \to \mathbb{R}$ be a positive convex function with $0 \leq a < b$ and $f \in L[a, b]$

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha}(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

with $\alpha > 0$.

The inequality (1.1) was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality was known as Hermite-Hadamard inequlity, because this inequality was found by Mitrinovic Hermite and Hadamard' note in Mathesis in 1974.

For several recent results concerning inequality (1.1), (1.2), see ([1]-[27]) where further references are listed.

Definition 1.1. [9] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ where $a, b \in I$ and $t \in [0, 1]$. The convex function is defined as

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b). \tag{1.3}$$

Definition 1.2. [11] Let $s \in (0, 1]$. A function $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \tag{1.4}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

Tunç and Yildirim in [27] introduced the following definition as follows:

Definition 1.3. A function $f : I \subseteq R \to R$ is said to belong to the class of $MT(I)$ if it is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the inequality;

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y).$$

Definition 1.4. [6] Let $\varphi : (0, 1) \to (0, \infty)$ be a measurable function. We say that the function $f : I \to [0, \infty)$ is a $\varphi-$convex function on the interval I if $x, y \in I$ we have

$$f(tx + (1-t)y) \leq t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y).$$

Remark 1.1. According to definition 4 for the special choose of φ we can obtain following;

If we take $\varphi(t) \equiv 1$, we obtain classical convex.

If we take $\varphi(t) = t^{s-1}$, we obtain $s-$convex.

If we take $\varphi(t) = \dfrac{1}{2\sqrt{t}\sqrt{1-t}}$, we obtain $MT-$convex.

Definition 1.5. [10] Let $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$. The quasi-convexity is defined as

$$f\left(tx + (1-t)y\right) \leq \max\left\{f(x), f(y)\right\}.$$

Now, we give some necessary definitions of fractional calculus theory which are used further in this paper.

Definition 1.6. [20] Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) dt, \; x > a \tag{1.5}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t) dt, \; x < b, \tag{1.6}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Samko et al. in [20] used following definitions as follows:

Definition 1.7. The Euler Beta function is defined as follows:

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \; x, y > 0.$$

The incomplate beta function is defined as follows:

$$\beta(a, x, y) = \int_0^a t^{x-1} (1-t)^{y-1} dt, \; x, y > 0, \; 0 < \alpha < 1.$$

Recently, in a series of research publications, Diaz et al. ([3],[4],[5]) have introduced k -gamma and k -beta functions and proved a number of their properties.

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{x}{k} 1}}{(x)_{n,k}} \tag{1.7}$$

$(x)_{n,k} = \prod_{j=0}^{n-1}(x+jk), k > 0$ is the Pochhammer k -symbols for factorial function. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k -gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{\frac{t^k}{k}} dt. \tag{1.8}$$

Clearly, $\Gamma(x) = \lim_{k \to 1} \Gamma_k(x)$, $\Gamma_k(x) = k^{\frac{x}{k}} \Gamma_k(\frac{x}{k})$ and $\Gamma_k(x+k) = x\Gamma_k(x)$.

Definition 1.8. [4] The k-Beta function is defined the following formula

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt, \; x, y, k > 0.$$

Remark 1.2. The following identity, connection the Γ_k, B_k function, is also given in [4]

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \; Re(x) > 0, \; Re(y) > 0.$$

k -gamma also leads to another interesting direction, k -fractional integral defined by

$$I_k^{\alpha}(f(x)) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt. \tag{1.9}$$

Remark 1.3. [23] When $k \to 1$, it then reduces to the classical Riemann-Liouville fractional integral

$$I^{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt. \tag{1.10}$$

Note that, $I_k^{\alpha} f$ exists in C_0 if $f \in C_0$, where C_0 be the class of all functions which are continuous and integrable on the interval $(0, \infty)$.

2 Main Results

Now, we give a useful lemma that will be used later.

Lemma 2.1. *Let $f\colon I \subset R \to R$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\frac{\alpha}{k} > 0$ we have:*

$$I_f\left(x, \lambda, \tfrac{\alpha}{k}; a, b\right) = \frac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)} \int_0^1 t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right) f''(tx+(1-t)a)\,dt$$

$$+\frac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)} \int_0^1 t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right) f''(tx+(1-t)b)\,dt,$$

where

$$I_f\left(x, \lambda, \tfrac{\alpha}{k}; a, b\right)$$

$$= (1-\lambda)\left[\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}\right] f(x) + \lambda\left[\frac{(x-a)^{\frac{\alpha}{k}} f(a)+(b-x)^{\frac{\alpha}{k}} f(b)}{b-a}\right]$$

$$+\left(\frac{1}{\frac{\alpha}{k}+1}-\lambda\right)\left[\frac{(b-x)^{\frac{\alpha}{k}+1}-(x-a)^{\frac{\alpha}{k}+1}}{b-a}\right] f'(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[{}_kJ^{\alpha}_{x-}f(a) + {}_kJ^{\alpha}_{x+}f(b)\right].$$

Proof. Integrating by parts and changing the variable, for $x \neq a$ we get

$$\begin{aligned}
&\int_0^1 t\left[\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right] f''(tx+(1-t)a)\,dt \\
&= \left[\left(\tfrac{\alpha}{k}+1\right)\lambda - 1\right]\frac{f'(x)}{x-a} + \frac{\left(\frac{\alpha}{k}+1\right)}{(x-a)^2}\left[(1-\lambda)f(x)+\lambda f(a)\right] \\
&- \frac{\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)}{(x-a)^{\frac{\alpha}{k}+2}}\int_a^x (u-a)^{\frac{\alpha}{k}-1} f(u)\,du \\
&= \left[\left(\tfrac{\alpha}{k}+1\right)\lambda - 1\right]\frac{f'(x)}{x-a} + \frac{\left(\frac{\alpha}{k}+1\right)}{(x-a)^{\frac{\alpha}{k}+2}}\left[(1-\lambda)f(x)+\lambda f(a)\right] \\
&- \frac{\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)}{(x-a)^{\frac{\alpha}{k}+2}}\Gamma_k(\alpha+k)\,{}_kJ^{\alpha}_{x-}f(a).
\end{aligned} \tag{2.1}$$

Similarly for $x \neq b$, we get

$$\begin{aligned}
&\int_0^1 t\left[\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right] f''(tx+(1-t)b)\,dt \\
&= \left[\left(\tfrac{\alpha}{k}+1\right)\lambda - 1\right]\frac{f'(x)}{x-b} + \frac{\left(\frac{\alpha}{k}+1\right)}{(b-x)^2}\left[\lambda f(b)-(\lambda-1)f(x)\right] \\
&- \frac{\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)}{(b-x)^{\frac{\alpha}{k}+2}}\int_a^x (u-b)^{\frac{\alpha}{k}-1} f(u)\,du \\
&= \left[\left(\tfrac{\alpha}{k}+1\right)\lambda - 1\right]\frac{f'(x)}{x-b} + \frac{\left(\frac{\alpha}{k}+1\right)}{(b-x)^2}\left[(1-\lambda)f(x)+\lambda f(a)\right] \\
&- \frac{\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)}{(b-x)^{\frac{\alpha}{k}+2}}\Gamma_k(\alpha+k)\,{}_kJ^{\alpha}_{x+}f(b).
\end{aligned} \tag{2.2}$$

Multiplying both sides of (2.1) and (2.2) by $\frac{(x-a)^{\frac{\alpha}{k}+2}}{(b-a)}$ and $\frac{(b-x)^{\frac{\alpha}{k}+2}}{(b-a)}$ respectively, and adding the resulting we obtain;

$$\frac{(x-a)^{\frac{\alpha}{k}+2}}{(b-a)}\int_0^1 t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right) f''(tx+(1-t)a)\,dt$$

$$+\frac{(b-x)^{\frac{\alpha}{k}+2}}{(b-a)}\int_0^1 t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right) f''(tx+(1-t)b)\,dt$$

$$= \left(1-\lambda\left(\tfrac{\alpha}{k}+1\right)\right)\left[\frac{(b-x)^{\frac{\alpha}{k}+1}-(x-a)^{\frac{\alpha}{k}+1}}{(b-a)}\right] f'(x) + \lambda\left(\tfrac{\alpha}{k}+1\right)\left[\frac{(b-x)^{\frac{\alpha}{k}+1}f(b)+(x-a)^{\frac{\alpha}{k}+1}f(a)}{(b-a)}\right]$$

$$+(1-\lambda)\left(\tfrac{\alpha}{k}+1\right)\left[\frac{(b-x)^{\frac{\alpha}{k}}-(x-a)^{\frac{\alpha}{k}}}{b-a}\right] f(x) - \frac{\Gamma_k(\alpha+k)\left(\frac{\alpha}{k}+1\right)}{b-a}\left[{}_kJ^{\alpha}_{x-}f(a) + {}_kJ^{\frac{\alpha}{k}}_{x+}f(b)\right].$$

□

Theorem 2.2. *Let $\varphi : (0,1) \to (0,\infty)$ be a measurable function. Assume also that $f : I \subset [0,\ \infty) \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I and $f'' \in L[a,b]$, where $a,b \in I^0$ with $a < b$. If $|f''|^q$ is $\varphi-$convex on $[a,b]$ for some fixed $q \geq 1$, then we get the following inequality for k-fractional integrals*

$$\begin{aligned}
&\left|I_f\left(x,\lambda,\tfrac{\alpha}{k},t,\varphi;a,b\right)\right| \\
&\leq A_1^{1-\frac{1}{q}}\left(\tfrac{\alpha}{k},\lambda\right)\left[\tfrac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{A_2(\tfrac{\alpha}{k},\lambda,t,\varphi)\left|f''(x)\right|^q + A_3\left(\tfrac{\alpha}{k},\lambda,t,\varphi\right)\left|f''(a)\right|^q\right\}^{\frac{1}{q}}\right. \\
&\left.+\tfrac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{A_2\left(\tfrac{\alpha}{k},\lambda,t,\varphi\right)\left|f''(x)\right|^q + A_3\left(\tfrac{\alpha}{k},\lambda,t,\varphi\right)\left|f''(b)\right|^q\right\}^{\frac{1}{q}}\right]
\end{aligned} \tag{2.3}$$

for any $x = ta + (1-t)b, t \in [0,1], \lambda \in [0,1]$, and $\frac{\alpha}{k} > 0$, where

$$\begin{aligned}
A_1\left(\tfrac{\alpha}{k},\lambda\right) &= \tfrac{\frac{\alpha}{k}\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{2}{\frac{\alpha}{k}}}+1}{\frac{\alpha}{k}+2} - \left(\tfrac{\alpha}{k}+1\right)\tfrac{\lambda}{2}, \\
A_2\left(\tfrac{\alpha}{k},\lambda,t,\varphi\right) &= \int_0^1\left|t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right)\right|t\varphi(t)\,dt, \\
A_3\left(\tfrac{\alpha}{k},\lambda,t,\varphi\right) &= \int_0^1\left|t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right)\right|(1-t)\varphi(1-t)\,dt.
\end{aligned}$$

Corollary 2.3. *In Theorem 1, if we choose $\varphi(t) = 1$, we get the following inequality*

$$\begin{aligned}
&\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right| \\
&\leq \left(\tfrac{\alpha\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{2k}{\alpha}}+k}{\alpha+2k} - \tfrac{\left(\frac{\alpha}{k}+1\right)\lambda}{2}\right)^{1-\frac{1}{q}} \\
&\times\left[\tfrac{(x-a)^{\alpha+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{\left(\frac{3-\left(\frac{\alpha}{k}+3\right)\left(\frac{\alpha}{k}+1\right)\lambda+2\frac{\alpha}{k}\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{3k}{\alpha}}}{3\left(\frac{\alpha}{k}+3\right)}\right)\left|f''(x)\right|\right.\right. \\
&\left.+\left(\frac{\alpha\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{2k}{\alpha}}}{\alpha+2k} - \frac{2k\lambda\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{3k}{\alpha}}}{3(\alpha+3k)} + \frac{\alpha\left(\frac{\alpha}{k}+1\right)\lambda}{6k} - \frac{\alpha k}{(\alpha+2k)(\alpha+3k)}\right)\left|f''(a)\right|\right\} \\
&+\tfrac{(b-x)^{\alpha+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{\left(\frac{3-\left(\frac{\alpha}{k}+3\right)\left(\frac{\alpha}{k}+1\right)\lambda+2\frac{\alpha}{k}\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{3k}{\alpha}}}{3\left(\frac{\alpha}{k}+3\right)}\right)\left|f''(x)\right|\right. \\
&\left.\left.+\left(\frac{\alpha\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{2k}{\alpha}}}{\alpha+2k} - \frac{2k\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{1+\frac{3k}{\alpha}}}{3(\alpha+3k)} + \frac{\alpha\left(\frac{\alpha}{k}+1\right)\lambda}{6k} - \frac{\alpha k}{(\alpha+2k)(\alpha+3k)}\right)\left|f''(b)\right|\right\}\right].
\end{aligned}$$

Corollary 2.4. *In Theorem 1, if we choose $\varphi(t) = t^{s-1}$, we obtain the following inequality:*

$$\begin{aligned}
&\left|I_f\left(x,\lambda,\tfrac{\alpha}{k},t,\varphi;a,b\right)\right| \\
&\leq A_1^{1-\frac{1}{q}}\left(\tfrac{\alpha}{k},\lambda\right)\left[\tfrac{(x-a)^{\frac{\alpha}{k}+2}}{b-a}\left\{\left|f''(x)\right|^q A_4\left(\tfrac{\alpha}{k},\lambda,s\right) + \left|f''(a)\right|^q A_5\left(\tfrac{\alpha}{k},\lambda,t\right)\right\}^{\frac{1}{q}}\right. \\
&\left.+\tfrac{(b-x)^{\frac{\alpha}{k}+2}}{b-a}\left\{\left|f''(x)\right|^q A_4\left(\tfrac{\alpha}{k},\lambda,s\right) + \left|f''(b)\right|^q A_5\left(\tfrac{\alpha}{k},\lambda,t\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

where

$$A_4\left(\tfrac{\alpha}{k},\lambda,s\right) = 2\frac{\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{\frac{(s+2)k}{\alpha}+1}}{s+2} - 2\frac{k\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{\frac{(s+2)k}{\alpha}+1}}{\alpha+ks+k2} + \frac{k}{\alpha+ks+k2}$$

$$A_5\left(\tfrac{\alpha}{k},\lambda,t\right) = \left(\tfrac{\alpha}{k}+1\right)\lambda\beta\left(\left(\left(\tfrac{\alpha}{k}+1\right)\lambda\right)^{\frac{k}{\alpha}},2,s+1\right) - \beta\left(\left(\left(\tfrac{\alpha}{k}+1\right)\lambda\right)^{\frac{k}{\alpha}},\alpha+2,s+1\right)$$
$$+\beta\left(1-\left(\left(\tfrac{\alpha}{k}+1\right)\lambda\right)^{\frac{k}{\alpha}},\alpha+2,s+1\right) - \left(\tfrac{\alpha}{k}+1\right)\lambda\beta\left(1-\left(\left(\tfrac{\alpha}{k}+1\right)\lambda\right)^{\frac{k}{\alpha}},2,s+1\right).$$

Theorem 2.5. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function. Assume also that $f: I\subset [0,\ \infty)\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval and $f''\in L[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''|^q$ is $\varphi-$convex on $[a,b]$ for some fixed $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get the following inequality for $k-$fractional integrals*

$$\left|I_f\left(x,\lambda,\tfrac{\alpha}{k},t,\varphi;a,b\right)\right|$$
$$\leq B^{\frac{1}{p}}\left(\tfrac{\alpha}{k},\lambda,p\right)\left[\frac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{\left(|f''(x)|^q+|f''(a)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right. \tag{2.4}$$
$$\left.+\frac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{\left(|f''(x)|^q+|f''(b)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right],$$

for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\frac{\alpha}{k}>0$, where${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a,b,c,z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

for $0<b<c$ and $|z|<1$, and

$$B\left(\tfrac{\alpha}{k},\lambda,p\right) = \frac{\left(\left(\frac{\alpha}{k}+1\right)\lambda\right)^{\frac{1+p+\frac{\alpha}{k}p}{\frac{\alpha}{k}}}}{\frac{\alpha}{k}}\left\{\Gamma(1+p)\Gamma\left(\frac{1+p+\frac{\alpha}{k}}{\frac{\alpha}{k}}\right)\ {}_2F_1\left(1,1+p,2+p+\frac{1+p}{\frac{\alpha}{k}},1\right)\right.$$
$$\left.+\beta\left(1+p,-\frac{1+p+\frac{\alpha}{k}p}{\frac{\alpha}{k}}\right)-\beta\left(\lambda,1+p,-\frac{1+p+\frac{\alpha}{k}p}{\frac{\alpha}{k}}\right)\right\}.$$

Corollary 2.6. *In Theorem 2 if we choose $\varphi(t)=1$, then we get the following inequality which holds for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$;*

$$\left|I_f\left(x,\lambda,\tfrac{\alpha}{k},t,\varphi;a,b\right)\right| \leq \left(\int_0^1\left|t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right)\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^{\frac{\alpha}{k}+2}}{b-a}\left\{\frac{\left(|f''(x)|^q+|f''(a)|^q\right)}{2}\right\}^{\frac{1}{q}}+\frac{(b-x)^{\frac{\alpha}{k}+2}}{b-a}\left\{\frac{\left(|f''(x)|^q+|f''(b)|^q\right)}{2}\right\}^{\frac{1}{q}}\right].$$

In Theorem 2 if we take $\varphi(t)=t^{s-1}$, then we can obtain the following inequality:

$$\left|I_f\left(x,\lambda,\tfrac{\alpha}{k},t,\varphi;a,b\right)\right| \leq \left(\int_0^1\left|t\left(\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right)\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^{\alpha+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{\frac{\left(|f''(x)|^q+|f''(a)|^q\right)}{s+1}\right\}^{\frac{1}{q}}+\frac{(b-x)^{\alpha+2}}{\left(\frac{\alpha}{k}+1\right)b-a}\left\{\frac{\left(|f''(x)|^q+|f''(b)|^q\right)}{s+1}\right\}^{\frac{1}{q}}\right].$$

From the definition of quasi-convex, we can obtain the following theorems for $k-$fractional integrals.

Theorem 2.7. *Let $f\colon I\subset R\to R$ be a twice differentiable function on I° and $f''\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f''|^{q}$ is quasi-convex on $[a,b]$ for some fixed $q\geq 1$, then we get the following inequality for $k-$fractional integrals with $x\in[a,b]$, $\lambda\in[0,1]$ and $\frac{\alpha}{k}>0$*

$$\begin{aligned}\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right| \leq D\left(\tfrac{\alpha}{k},\lambda\right)\Bigg\{ & \tfrac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\max\left\{\left|f''(x)\right|^{q},\left|f''(a)\right|^{q}\right\}\right)^{\frac{1}{q}} \\ & +\tfrac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\max\left\{\left|f''(x)\right|^{q},\left|f''(b)\right|^{q}\right\}\right)^{\frac{1}{q}}\Bigg\},\end{aligned} \tag{2.5}$$

where

$$D\left(\frac{\alpha}{k},\lambda\right)=\begin{cases}\frac{\alpha\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{\alpha+2k}{\alpha}}+k}{\alpha+2k}-\frac{\left(\frac{\alpha}{k}+1\right)\lambda}{2} & ,0\leq\lambda\leq\frac{1}{\frac{\alpha}{k}+1}\\ \frac{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)\lambda-2}{2\left(\frac{\alpha}{k}+2\right)} & ,\frac{1}{\frac{\alpha}{k}+1}<\lambda\leq 1.\end{cases}$$

Proof. By property of the modulus, Lemma 1 and power-mean inequality we get

$$\begin{aligned}\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right| \leq & \tfrac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|\left|f''(tx+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}} \\ & +\tfrac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\int_0^1 t\left|t\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}} \\ & \times(\int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|\left|f''(tx+(1-t)b)\right|^{q}dt)^{\frac{1}{q}}.\end{aligned}$$

Since $|f''|^{q}$ is quasi-convex on $[a,b]$ we get

$$\begin{aligned}&\int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|\left|f''(tx+(1-t)a)\right|^{q}dt \\ &\leq D\left(\tfrac{\alpha}{k},\lambda\right)\max\{\left|f''(x)\right|^{q},\left|f''(a)\right|^{q}\},\end{aligned}$$

And similarly we get

$$\begin{aligned}&\int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|\left|f''(tx+(1-t)b)\right|^{q}dt \\ &\leq D\left(\tfrac{\alpha}{k},\lambda\right)\max\{\left|f''(x)\right|^{q},\left|f''(b)\right|^{q}\},\end{aligned}$$

where we use the fact that

$$\begin{aligned}D\left(\tfrac{\alpha}{k},\lambda\right) &= \int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|dt \\ &= \begin{cases}\left(\frac{\alpha}{k}+1\right)\lambda\int_0^{\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{k}{\alpha}}}t\,dt-\int_0^{\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{k}{\alpha}}}t^{\frac{\alpha}{k}+1}dt & \\ -\left(\frac{\alpha}{k}+1\right)\lambda\int_{\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{k}{\alpha}}}^{1}t\,dt+\int_{\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{k}{\alpha}}}^{1}t^{\frac{\alpha}{k}+1}dt & ,0\leq\lambda\leq\frac{1}{\frac{\alpha}{k}+1}\\ \left(\frac{\alpha}{k}+1\right)\lambda\int_0^1 t\,dt-\int_0^1 t^{\frac{\alpha}{k}+1}dt & ,\frac{1}{\frac{\alpha}{k}+1}<\lambda\leq 1\end{cases} \\ &= \begin{cases}\frac{\alpha\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{\alpha+2k}{\alpha}}+k}{\alpha+2k}-\frac{\left(\frac{\alpha}{k}+1\right)\lambda}{2} & ,0\leq\lambda\leq\frac{1}{\frac{\alpha}{k}+1}\\ \frac{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)\lambda-2}{2\left(\frac{\alpha}{k}+2\right)} & ,\frac{1}{\frac{\alpha}{k}+1}<\lambda\leq 1.\end{cases}\end{aligned}$$

This completes the proof. □

Corollary 2.8. *In Theorem 3, if we take $q = 1$, then we obtain the following inequality*

$$\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right| \le \left(\int_0^1 t\left|\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right|dt\right)$$

$$\times\left[\frac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\max\left\{\left|f''(x)\right|,\left|f''(a)\right|\right\} + \frac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\max\left\{\left|f''(x)\right|,\left|f''(b)\right|\right\}\right].$$

Corollary 2.9. *In Theorem 3, if choose $x = \frac{a+b}{2}$ and $k = 1$, we can obtain corollary 2.2 in [10].*

Corollary 2.10. *If we choose $x = \frac{a+b}{2}$ and $k = 1$ in Theorem 3, we can obtain the corollary 2.3, 2.4, 2.5 in [10], respectively for $\lambda = \frac{1}{3}$, $\lambda = 0$, $\lambda = 1$.*

Theorem 2.11. *Let $f\colon I \subset R \to R$ be a twice differentiable function on I° and $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for some fixed $q > 1$, then we can obtain the following inequality for k-fractional integrals with $x \in [a,b]$, $\lambda \in [0,1]$ and $\frac{\alpha}{k} > 0$*

$$\begin{aligned}\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right| \le D^{\frac{1}{p}}\left(\tfrac{\alpha}{k},\lambda,p\right)&\left[\frac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\frac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right],\end{aligned} \tag{2.6}$$

where $p = \frac{q}{q-1}$,

$$D\left(\frac{\alpha}{k},\lambda,p\right) = \begin{cases} \frac{k}{p\alpha+p+k} & ,\lambda = 0 \\ \left[\frac{k\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{p\alpha+p+k}{\alpha}}}{\alpha}\beta\left(\frac{k(p+1)}{\alpha},p+1\right)\right. & \\ \left.+\frac{[k-\alpha\lambda-\lambda]^{p+1}}{\alpha(p+1)}\right]\ {}_2F_1\left(\frac{\alpha-k-kp}{\alpha},1,p+2;\frac{k-\alpha\lambda-k\lambda}{k}\right) & ,0 < \lambda \le \frac{1}{\frac{\alpha}{k}+1} \\ \frac{[(\alpha+k)\lambda]^{\frac{(\alpha+k)p+1}{\alpha}}}{\alpha}\beta\left(\frac{k}{(\alpha+k)\lambda};\frac{k(p+1)}{\alpha},p+1\right) & \frac{1}{\frac{\alpha}{k}+1} < \lambda \le 1 \end{cases}$$

${}_2F_1$ is Hypergeometric function.

Proof. By property of the modulus, Lemma 1 and using the Hölder inequality we get

$$\begin{aligned}&\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right|\\ &\le \frac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\int_0^1 t\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\left|f''(tx+(1-t)a)\right|dt\\ &+\frac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\int_0^1 t\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\left|f''(tx+(1-t)b)\right|dt\\ &\le \frac{(x-a)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\int_0^1 t^p\left|\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)(b-a)}\left(\int_0^1 t^p\left|\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

Since $|f''|^q$ is quasi-convex on $[a,b]$ we get

$$\int_0^1 \left|f''(tx+(1-t)a)\right|^q dt \leq \max\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q\}$$

$$\int_0^1 \left|f''(tx+(1-t)b)\right|^q dt \leq \max\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q\}$$

and

$$D\left(\tfrac{\alpha}{k},\lambda,p\right)=\int_0^1 t^p\left|\left(\tfrac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right|^p dt$$

$$=\begin{cases} \int_0^1 t^{\left(\frac{\alpha}{k}+1\right)p}dt & ,\lambda=0 \\ \left[\int_0^{\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{k}{\alpha}}} t^p\left[\left(\frac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right]^p dt\right. & \\ \left.+\int_{\left[\left(\frac{\alpha}{k}+1\right)\lambda\right]^{\frac{k}{\alpha}}}^1 t^p\left[t^{\frac{\alpha}{k}}-\left(\frac{\alpha}{k}+1\right)\lambda\right]^p dt\right] & ,0<\lambda\leq\frac{1}{\frac{\alpha}{k}+1} \\ \int_0^1 t^p\left[\left(\frac{\alpha}{k}+1\right)\lambda - t^{\frac{\alpha}{k}}\right]^p dt & ,\frac{1}{\frac{\alpha}{k}+1}<\lambda\leq 1 \end{cases}$$

This completes the proof. □

Corollary 2.12. *If we choose $x=\frac{a+b}{2}$ in Theorem 4, we get*

$$\left|I_f\left(x,\lambda,\tfrac{\alpha}{k};a,b\right)\right|$$

$$\leq\left(\int_0^1 t^p\left|\left(\tfrac{\alpha}{k}+1\right)\lambda-t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}\frac{(b-a)^{\frac{\alpha}{k}+1}}{\left(\frac{\alpha}{k}+1\right)}\left\{\left(\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\},$$

Corollary 2.13. *If we choose $x=\frac{a+b}{2}$ and $k=1$ in Theorem 4, we can obtain the corollary 2.7, 2.9, 2.10 in [10], respectively for $\lambda=\frac{1}{3}$, $\lambda=0$, $\lambda=1$.*

3 Conclusion

In the present paper, we construct some new inequalities of the Simpson-like and the Hadamard-like type for functions whose second derivatives are φ-convex and quasi-convex via the integral equality which is given in this work.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Beckenbach EF. Convex functions. Bull. Amer. Math. Soc. 1948;54:439-460.
Available: http://dx.doi.org/10.1090/s0002-9904-1948-08994-7

[2] Dahmani Z. On Minkowski and Hermite-Hadamard integral inequalities via fractional integration. Ann. Funct. Anal. 2010;1(1):51-58.
Available: http://dx.doi.org/10.15352/afa/1399900993

[3] Diaz R, Teruel C. q,k -Generalized gamma and beta functions. J. Nonlinear Math. Phys. 2005;12:118-134.

[4] Diaz R, Pariguan E. On hypergeometric functions and Pochhammer k -symbol. Divulg. Mat. 2007;15:179-192.

[5] Diaz R, Ortiz C, Pariguan E. On the k -gamma q -distribution. Cent. Eur. J. Math. 2010;8:448-458.

[6] Dragomir SS. Inequalities of Jensen type for φ-convex functions. Fasc. Math. 2015;55:35-52.

[7] Hudzik H, Maligranda L. Some remarks on s-convex functions. Aequationes Math. 1994;48(1):100-111.

[8] Işcan I, Bekar K, Numan S. Hermite-Hadamard an Simpson type inequalities for differentiable quasi-geometrically convex functions. Turkish J. of Anal. and Number Theory. 2014;2(2):42-46.
Available: http://dx.doi.org/10.12691/tjant-2-2-3

[9] Işcan I. New estimates on generalization of some integral inequalities for ds-convex functions and their applications. Int. J. Pure Appl. Math. 2013;86(4):727-746.
Available: http://dx.doi.org/10.12732/ijpam.v86i4.11

[10] Işcan I. Generalization of different type integral inequalities via fractional integrals for functions whose second derivatives absolute value are quasi-convex. Konuralp Journal of Mathematics. 2013;1(2):67-79.

[11] Işcan I. On generalization of different type integral inequalities for s-convex functions via fractional integrals. Mathematical Sciences and Applications E-Notes. 2014;2(1):55-67.

[12] Kavurmaci H, Avci M, Özdemir ME. New inequalities of Hermite- Hadamard's type for convex functions with applications. Journ. of In-equal. and Appl. 2011;2011:86.
Available: http://dx.doi.org/10.1186/1029-242x-2011-86

[13] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. North-Holland Mathematical Studies. Elsevier, Amsterdam. 2006;204.

[14] Kokologiannaki CG. Properties and inequalities of generalized k-gamma, beta and zeta functions. Int. J. Contemp. Math. Sciences. 2010;5:653-660.

[15] Krasniqi V. A limit for the k -gamma and k -beta function. Int. Math. Forum. 2010;5:1613-1617.

[16] Mansour M. Determining the k -generalized gamma function Γ_k(x) by functional equations. Int. J. Contemp. Math. Sciences. 2009;4:1037-1042.

[17] Mihesan VG. A generalization of the Convexity. Seminar on Functional Equation Approx. and Convex. Cluj-Napoca, Romania; 1993.

[18] Özdemir ME, Avci M, Kavurmaci H. Hermite-Hadamard type inequal- ities for s-convex and s-concave functions via fractional integrals. ArXiv:1202.0380v1[math.CA]; 2012.

[19] Park J. On some integral inequalities for twice differentiable quasi-convex and convex functions via fractional integrals. Applied Mathematical Sciences. HIKARI Ltd, www.m-hikari.com. 2015;9(62):3057-3069.
Available: http://dx.doi.org/10.12988/ams.2015.53248

[20] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives. Theory and Applications, Gordon and Breach. 1993;ISBN 2881248640.

[21] Sarikaya MZ, Ogunmez H. On new inequalities via Riemann-Liouville fractional integration. Abstract and Applied Analysis. 2012;2012:10 pages, Art ID:428983.
Available: http://dx.doi.org/10.1155/2012/428983

[22] Sarikaya MZ, Set E, Yildiz Basak H. N. Hermite- Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. and Comput. Model; 2011.
Available: http://dx.doi.org/10.1016/j.mcm.2011.12.048

[23] Set E, Tomar M, Sarikaya MZ. On generalized Grüss type inequalities for $k-$fractional integrals. Applied Mathematics and Computation. 2015;269:29-34.

[24] Set E, Sarikaya MZ, Özdemir ME. Some Ostrowski's type inequalities for functions whose second derivatives are s-convex in the second sense. ArXiv:1 006.24 88v1 [math. CA]; 2010.

[25] Toader Gh. On a generalization of the convexity. Mathematica. 1988;30(53):83-87.

[26] Tunc M. On some new inequalities for convex functions. Turk. J. Math. 2011;35:1-7.

[27] Tunc M, Yildirim H. On MT-convexity. ArXiv: 1205.5453 [math. CA]; 2012.

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