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Diffusion - Convection Equation of Galactic Cosmic Rays (GCR) in the Atmosphere and Its Analytical, Numerical Solutions by Using Finite Elements Method Using Parker Transport Equation

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This study depicts that to find the Analytical and numerical solutions of Diffusion – Convections Equations of Galactic Cosmic Rays (GCRs) by Finite Element method (FEM) and also to find the Energy Equation of GCRs by using a part of Parker's transport equation. This considers moreover centres on the exactness and acknowledgment of the FEM strategy by utilizing dissemination blunder, scattering mistake and add up to blunder investigation. The comes about are depicted both graphically and in a unthinkable frame, which

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essentially guarantees the method's legitimacy and the algorithm's proficiency to maintain the exactness, effortlessness, and nonlinear Convection-Diffusion Equation conditions. The proposed method may be connected for tackling any nonlinear convection diffusion equation. We concentrate on assaying the confluence and stability of the nonlinear parabolic partial differential equation. This study focuses on the delicacy and acceptance of FEM method by exercising dispersion error dissipation error, and total error analysis.

Keywords: Analytical solutions; numerical solutions; finite element method; convection-diffusion equation; Parker's transport equation.

1 Introduction

Enormous beams of GCR are high energy particles or clusters of particles that are move through space at about the speed of light. They start from the Sun, from exterior of the Sun powered framework in our claim system, and form removed galaxies. Modelling authentic life and industrial quandaries by applying partial differential equations (PDEs) is challenging for researchers and scientists. Bosen.G [1] stated, a considerable number of quandaries arise from modelling nonlinear systems of differential equations. Researchers especially Kumar.A, et al. [2] have endeavoured to solve these quandaries analytically or numerically utilizing different methods, Leonard B.P. [3,4] and equations to obtain higher precision levels. Present study addresses the one-dimensional Convection-Diffusion equation, Dehghan M, [5], as this is a meaningful test to construct a novel discrete plan. As the base of Parker transport equation, we can evaluate the Analytical and Numerical solutions of Diffusion according to Pérez Guérrero JS [6,7] and Convection equation of Galactic Cosmic Rays, Basdevant C et al. [8]. Convection-diffusion equations, for instance, can represent real-world issues [9-14]. Because of the CD equation's significance, numerous researchers have developed various numerical techniques. Its prospective applications have drawn a lot of interest. The Finite Element Method (FEM) offers the most precise solutions to linear and nonlinear CD problems as well as parabolic equations among the methods listed above [15-17]. Convection, diffusion, and reaction are significant because they may be used to explain a variety of physical issues, including how the three processes affect how the concentration of one or more chemicals distributed in a medium changes [18-22]. Convection shows how substances move as a result of the transport medium whereas reaction is contact.

The Parker Transport Equation, Sadiq Akter Lima et al. [23] is to be written as

$$\frac{\partial \emptyset}{\partial t} + w.\frac{\partial \emptyset}{\partial x} = \mu \frac{\partial^2 \emptyset}{\partial x^2} + \frac{P}{3}\frac{\partial w}{\partial x}\frac{\partial \emptyset}{\partial P} + \frac{1}{P^2}\frac{\partial}{\partial P}\left(P^2 C_{PP}\frac{\partial \emptyset}{\partial P}\right) + \varepsilon_0 \sigma(x) \tag{1}$$

Now, from the above equation we consider the convection- diffusion (C-D) terms based on C. Zoppou [24] and solve the analytical solution for Convection Diffusion (C-D) equation of GCR, Kumar.A, et al. [2]

$$\frac{\partial \emptyset}{\partial t} = \mu \frac{\partial^2 \emptyset}{\partial x^2} - w.\frac{\partial \emptyset}{\partial x} \tag{2}$$

$t > 0$

$0 < x < K$

Where μ is diffusion coefficient and w is the GCR velocity in the x-direction. Now by applying boundary conditions

$$\emptyset(0, x) = 0$$

$$\emptyset(t, K) = 1 \tag{3}$$

By applying Initial conditions

$$\emptyset(0,\mathrm{x}) = \begin{cases} 0, & 0 \le x \le K \\ 1, & x = K \end{cases} \tag{4}$$

At present convert the partial differential equation (PDE) to pure PDE by transformation method

$$\emptyset(t,x) = D(t,x).u(t,x) \tag{5}$$

Substitute the equation (5) in equation (1) we get

$$D_t u + D u_t = \mu[D_{xx}u + 2D_x u_x + D u_{xx}] - w(D_x u + D u_x) \tag{6}$$

Dividing by D, we get

$$u_t = \mu u_{xx} + \mu u\left(\frac{D_{xx} - \frac{D_t}{\mu} - wD_x}{D}\right) - u_x\left(\frac{2\mu D_x - wD}{D}\right) \tag{7}$$

To get the pure diffusion PDE we require

$$\mu u\left(\frac{D_{xx} - \frac{D_t}{\mu} - wD_x}{D}\right) = 0 \tag{8}$$

$$u_x\left(\frac{2D_{xx} - wD}{D}\right) = 0 \tag{9}$$

$$D_x - \frac{w}{2\mu}D = 0 \tag{10}$$

The equation (10) has solution

$$D(t,x) = C(t).e^{\frac{w}{2\mu}x} \tag{11}$$

Substitute the equation (10) in (8) we get $\quad C(t) = C_1 e^{\frac{w^2}{4\mu}t}$ (12)

C_1 is a constant and we consider its value is 1 then we get

$$C(t) = e^{\frac{w^2}{4\mu}t} \tag{13}$$

Substitute equation (13) in (11)

$$D(t,x) = e^{(\frac{w^2 t}{4\mu} + \frac{wx}{2\mu})} \tag{14}$$

By using equation (14) which gives PDE to solve

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \tag{15}$$

By applying boundary condition from $\emptyset$ to u which leads to

$$\emptyset(t,0) = 0$$

$$D(t,0).u(t,0) = 0$$

$$e^{\frac{w^2 t}{4\mu}} u(t,0) = 0$$

$$u(t,0) = 0 \tag{16}$$

and also consider the conditions

$$\emptyset(t, K) = 1$$

$$\emptyset(t, K). u(t, K) = 1$$

$$e^{\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)}. u(t, K) = 1$$

$$u(t, K) = e^{-\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)} \tag{17}$$

Applying initial conditions

$$\emptyset(0, x) = \begin{cases} 0 & 0 \leq x \leq K \\ 1 & x = K \end{cases}$$

$$\emptyset(0, x). u(0, x) = \begin{cases} 0 & 0 \leq x \leq K \\ 1 & x = K \end{cases}$$

$$e^{\frac{wx}{2\mu}}. u(0, x) = \begin{cases} 0 & 0 \leq x \leq K \\ 1 & x = K \end{cases}$$

$$u(0, x) = \begin{cases} 0 & 0 \leq x \leq K \\ e^{-\frac{wx}{2\mu}} & x = K \end{cases} \tag{18}$$

By using variable separable method, the equation (15) become homogenous then let us consider

$$u(t, x) = \emptyset(t, x) + u_A(t, x) \tag{19}$$

Where u_A is the steady state solution and $\emptyset(t, x)$ satisfies the PDE with boundary conditions

$$u(t, x) = \emptyset(t, x) + \frac{x}{K} e^{-\left(\frac{w^2 t}{4\mu}+\frac{wK}{2\mu}\right)} \tag{20}$$

Substitute equation (20) in (15) we get

$$\emptyset_t = \mu \emptyset_{xx} + P(t, x) \tag{21}$$

This PDE with homogeneous boundary conditions along source term is

$$P(t, x) = -\frac{d}{dt} u_A(t, x) \tag{22}$$

Now our intension is to find the value of $\emptyset(t, x)$, as we know the solution to diffusion is given by the following eigen function expansion

$$\emptyset(t, x) = \sum_{n=1}^{\infty} \beta_n(t) \sin\left(\sqrt{\gamma_n} x\right) \tag{23}$$

Where $\gamma_n = (\frac{n\pi}{K})^2$ are eigen values for n =1, 2, . . . and sin $(\sqrt{\gamma_n} x)$ are eigen function. Substitute equation (23) in equation (21) in order to get an Ordinary Differential Equation to solve for $\beta_n(t)$ which gives

$$\sum_{n=1}^{\infty} \beta_n^{I}(t) \sin\left(\sqrt{\gamma_n} x\right) = \mu \sum_{n=1}^{\infty} - \beta_n(t) \gamma_n \sin\left(\sqrt{\gamma_n} x\right) + P(t, x) \tag{24}$$

Now we expand $P(t, x)$ in favour of eigen functions

$$P(t, x) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\sqrt{\gamma_n} x\right) \tag{25}$$

By applying orthogonality, we get

$$\int_0^K P(t,x).\sin\left(\sqrt{\gamma_n}x\right)dx = q_n(t).\frac{K}{2} \tag{26}$$

But

$$\int_0^K P(t,x).\sin\left(\sqrt{\gamma_n}x\right)dx = \frac{(-1)^n w^2 e^{-\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)}}{4\mu\sqrt{\gamma_n}} \tag{27}$$

From (26) we can find

$$q_n(t) = \frac{(-1)^n w^2 e^{-\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)}}{2K\mu\sqrt{\gamma_n}} \tag{28}$$

$$\beta_n^I(t) + \mu\gamma_n\,\beta_n(t) = q_n(t) \tag{29}$$

To solve the above equation with the integrating factor $\sigma = e^{\mu\gamma_n t}$ then

$$\beta_n(t) = \int_0^t q_n(\tau)\,.\,e^{\mu\gamma_n(\tau-t)}d\tau + C_n.\,e^{\mu\gamma_n t}e^{-\mu\gamma_n t} \tag{30}$$

Put equation (30) in (20) we lead to

$$u(t,x) = \frac{x}{K}e^{-\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)} + \sum_{n=1}^{\infty}\left[\int_0^t q_n(\tau)\,.\,e^{\mu\gamma_n(\tau-t)}d\tau + C_n.\,e^{-\mu\gamma_n t}\right]\sin\left(\sqrt{\gamma_n}x\right) \tag{31}$$

By applying initial conditions t =0, the equation (31) becomes

$$u(0,x) - \frac{x}{K}e^{-\left(\frac{wK}{2\mu}\right)} = \sum_{n=1}^{\infty} C_n \sin\left(\sqrt{\gamma_n}x\right) \tag{32}$$

Applying orthogonality

$$\int_0^K u(0,x).\sin\left(\sqrt{\gamma_n}x\right)dx - \int_0^K \frac{x}{K}e^{-\left(\frac{wK}{2\mu}\right)}\sin\left(\sqrt{\gamma_n}x\right)dx = C_n.\frac{K}{2} \tag{33}$$

Since $u(0,x) = 0$ then C_n value becomes

$$C_n = \frac{2}{K}\,.\frac{(-1)^n e^{-\left(\frac{wK}{2\mu}\right)}}{\sqrt{\gamma_n}} \tag{34}$$

But

$$\int_0^t q_n(\tau)\,.\,e^{\mu\gamma_n(\tau-t)}d\tau = \frac{2(-1)^n w^2 e^{-\left(\mu\gamma_n t+\frac{wK}{2\mu}\right)}\left(e^{\mu\gamma_n t-\frac{tw^2}{4\mu}}-1\right)}{n\pi(4\mu^2\gamma_n - w^2)} \tag{35}$$

$$u(t,x) = \frac{x}{K}e^{-\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n w^2 e^{-\left(\mu\gamma_n t+\frac{wK}{2\mu}\right)}\left(e^{\mu\gamma_n t-\frac{tw^2}{4\mu}}-1\right)}{n\pi(4\mu^2\gamma_n - w^2)} + \frac{2}{K}\,.\frac{(-1)^n e^{-\left(\frac{wK}{2\mu}\right)}}{\sqrt{\gamma_n}}\right].\,sin\left(\sqrt{\gamma_n}x\right) \tag{36}$$

Now we covert to $\emptyset(t,x)$, the final solution becomes

$$\emptyset(t,x) = e^{\left(\frac{w^2t}{4\mu}+\frac{wK}{2\mu}\right)}.\,u(t,x) \tag{37}$$

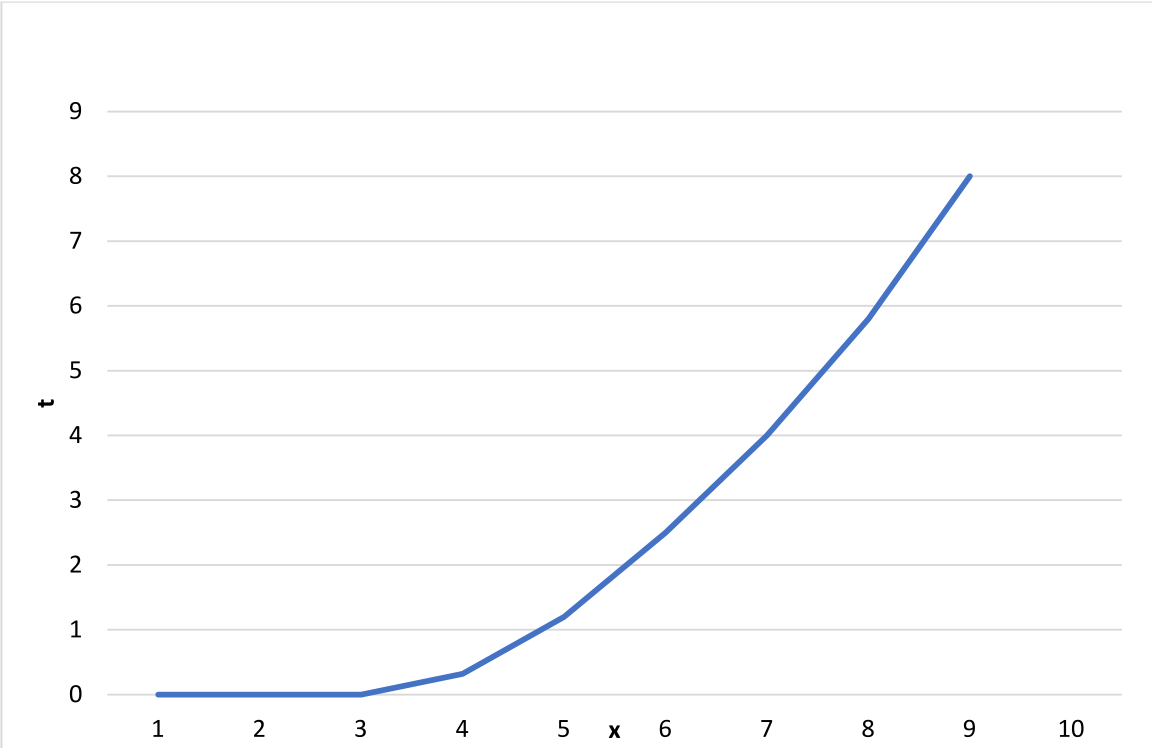

Fig. 1. Analytical solution of convection-diffusion equation at 1st second

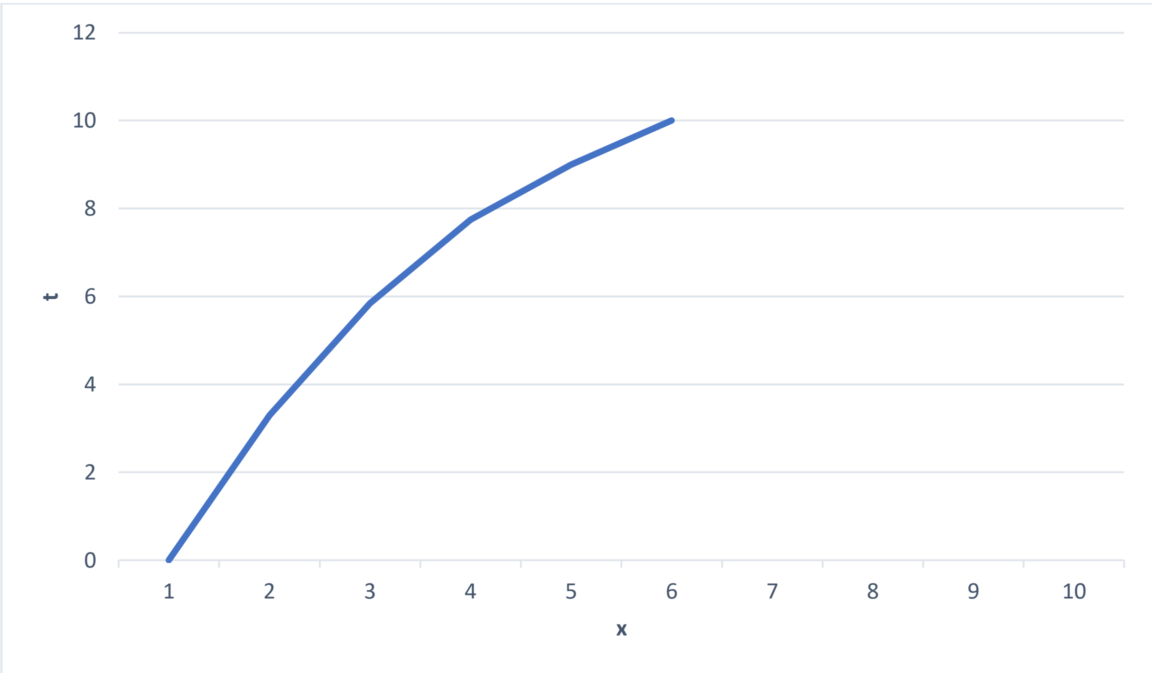

Fig. 2. Analytical solution of convection-diffusion equation at last (30th) second

2 The Equation of Energy

By considering the law conservation of kinetic energy and integrating under the given limits

$$\int_0^1 [\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) + w\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) dx = \mu \int_0^1 \emptyset \frac{\partial^2 \emptyset}{\partial x^2} dx \quad \int_0^1 [\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) + w\frac{\partial}{\partial x}\left(\frac{\emptyset^2}{2}\right) dx = -\mu \int_0^1 \left(\frac{\partial \emptyset}{\partial x}\right)^2 dx \quad (38)$$

To evaluate the last term by using integral by parts then the energy equation reduces to

$$\frac{d}{dt}\int_0^1\left(\frac{\emptyset^2}{2}\right)dx = -\,\mu\int_0^1\left(\frac{\partial\emptyset}{\partial x}\right)^2 dx \tag{39}$$

This is the energy decay associated with diffusion, the time needed to reach zero velocity at each and every limit so integrate equation (2)

$$\int_0^1\left(\frac{\partial\emptyset}{\partial t}+w\frac{\partial\emptyset}{\partial x}\right)dx = w\int_0^1\frac{\partial^2\emptyset}{\partial x^2}dx \tag{40}$$

Which results to the equation

$$\frac{d}{dt}\int_0^1\emptyset dx = \frac{dU}{dt} = \mu[\frac{\partial\emptyset}{\partial x}(1,t)-\frac{\partial\emptyset}{\partial x}(0,t)] \tag{41}$$

U represents the average velocity. Successively we get

$$\text{U(t)} = \mu\int_0^t[\frac{\partial\emptyset}{\partial x}(1,t)-\frac{\partial\emptyset}{\partial x}(0,t)]dt \tag{42}$$

3 Finite Element Methods

The Convection – Diffusion equation defined by Abdelkader Mojtabi [25], takes the form of

$$\frac{\partial\emptyset}{\partial t} = \mu\frac{\partial^2\emptyset}{\partial x^2} - w.\frac{\partial\emptyset}{\partial x} + f(C(x,t)),$$

$$(x, t) \in B \equiv \Lambda(0,T], T > 0 \tag{43}$$

Where f(∅) is the equation source, by setting $\mu = 1$, $w = 1$ and f(∅) = - ∅ in the equation(43) with the domain $\Lambda \in [0,5]$, we get numerical simulation as follows

$$\frac{\partial\emptyset}{\partial t} - \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial\emptyset}{\partial x} = -\,\emptyset\,,$$

$$(x,t) \in B \equiv [0,5] \times (0,T], T > 0 \tag{44}$$

Along with this, by applying the boundary and initial condition we come across the equations (45 and (46) respectively

$$\emptyset(x,0) = e^{-x}, x \in [0,5] \tag{45}$$

$$\emptyset(0,t) = e^{t},$$

$$\emptyset(5,t) = e^{t-5},$$

$$t \in [0,T], \qquad x \in \partial B \tag{46}$$

The analytical solution of the Diffusion – Convection equation is

$$\emptyset(x,t) = e^{t-x} \tag{47}$$

The trial solution of equation (43) is

$$\emptyset(x,0) = \sum_{i=1}^{n} w(t).\Psi_j(x) \tag{48}$$

Here, comparing exact solution with approximate solutions of equation (44). In that case, Finite number of elements [is n = 5 and two linear shape functions are represented by

$$l_1(\xi) = \frac{1-\xi}{2},\ l_2(\xi) = \frac{1+\xi}{2};\ \xi \in [-1,+1] \tag{49}$$

The convenient matrix form of equation (49) is given by

$$Z_{i,j} = \alpha_{i,j} + \beta_{i,j} + \delta_{i,j};$$

$$S_{i,j} = \int \Psi_i(x)[\textstyle\sum_{j=1}^{n} \Psi_j(x)]dx$$

$$\alpha_{i,j} = \int \frac{d\Psi_i(x)}{dx}.\frac{d\Psi_j(x)}{dx}\,dx,\ \text{F(t)} = [\,\frac{\partial \acute{\phi}}{\partial x}\Psi_j(x)]\ \ \beta_{i,j} = \int \Psi_i(x)[\textstyle\sum_{j=1}^{n} \frac{d\Psi_j(x)}{dx}]\,dx$$

$$\delta_{i,j} = \int \Psi_i(x)[\textstyle\sum_{j=1}^{n} \Psi_j(x)]dx$$

We obtain approximate results at regular intervals in spatial distribution by numerical computation.

Table 1. Exact and approximate solutions of (44) at h =Δt = 0.001

x	Exact	Finite Element Method (FEM)	Error
0.0	1.0010	0.9903	1.0×10^{-2}
0.5	0.6846	0.6811	3.5×10^{-3}
1.0	0.3682	0.3719	3.6×10^{-3}
1.5	0.2519	0.2533	1.4×10^{-3}
2.0	0.1355	0.1348	7.0×10^{-4}
2.5	0.0922	0.0925	3.0×10^{-4}
3.0	0.0498	0.0501	3.0×10^{-6}
3.5	0.0341	0.0342	1.0×10^{-10}
4.0	0.0183	0.0183	1.0×10^{-10}
4.5	0.0126	0.0126	1.0×10^{-10}
5.0	0.0068	0.0068	1.0×10^{-10}

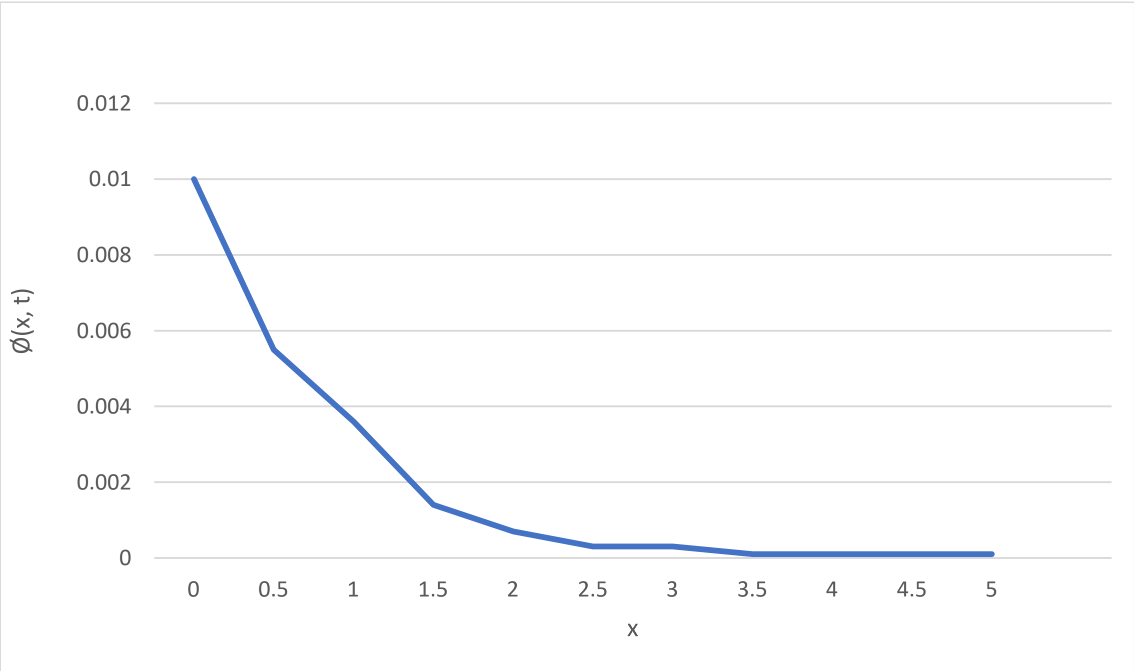

Fig. 3. Graph of exact and FEM solutions for equation (44)

Table 2. Exact and approximate solutions of (44) at h =Δt = 0.01

x	Exact	Finite Element Method (FEM)	Error
0.0	1.0101	0.9026	10.0×10^{-2}
0.5	0.6909	0.6553	2.8×10^{-2}
1.0	0.3716	0.4079	3.6×10^{-2}
1.5	0.2542	0.2688	5.3×10^{-2}
2.0	0.1367	0.1297	7.0×10^{-3}
2.5	0.0935	0.0915	5.0×10^{-3}
3.0	0.0503	0.0532	2.9×10^{-3}
3.5	0.0344	0.0357	3.4×10^{-4}
4.0	0.0185	0.0181	4.0×10^{-4}
4.5	0.0127	0.0126	3.0×10^{-4}
5.0	0.0068	0.0070	2.0×10^{-4}

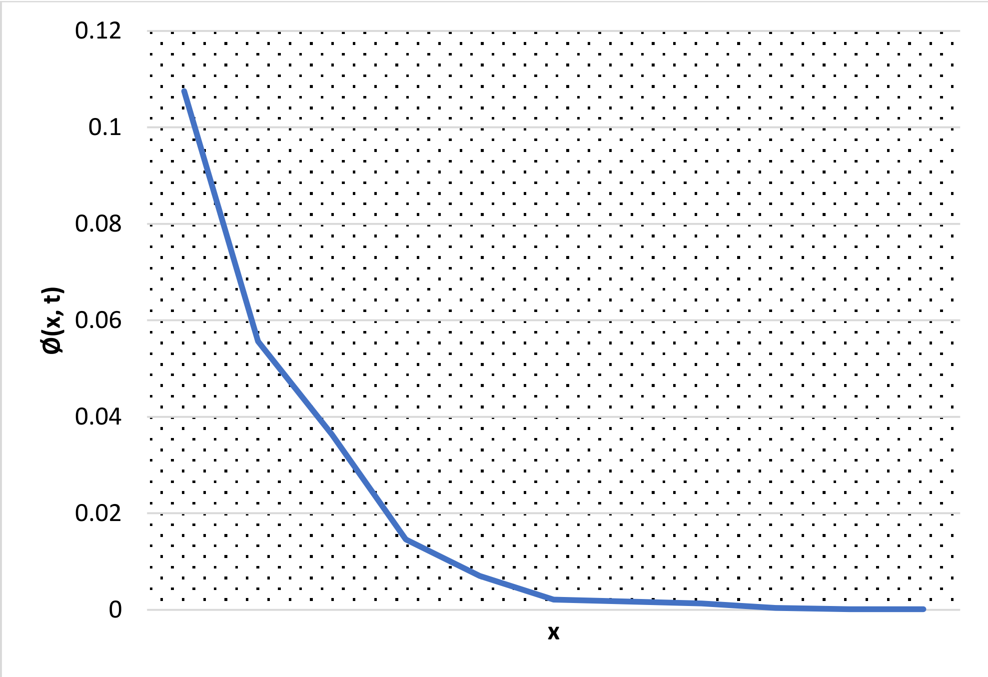

Fig. 4. Graph of exact and FEM solutions for equation (44)

Table 3. Exact and Approximate solutions of (44) at h =Δt = 0.05

x	Exact	Finite Element Method (FEM)	Error
0.0	1.0200	0.8052	2.1×10^{-1}
0.5	0.6977	0.6265	5.6×10^{-2}
1.0	0.3753	0.4478	7.2×10^{-2}
1.5	0.2067	0.2860	7.0×10^{-2}
2.0	0.1381	0.1241	1.4×10^{-3}
2.5	0.0945	0.0903	4.2×10^{-3}
3.0	0.0508	0.0565	5.8×10^{-3}
3.5	0.0348	0.0372	2.4×10^{-3}
4.0	0.0187	0.0179	8.0×10^{-4}
4.5	0.0128	0.0125	1.3×10^{-4}
5.0	0.0069	0.0074	5.0×10^{-4}

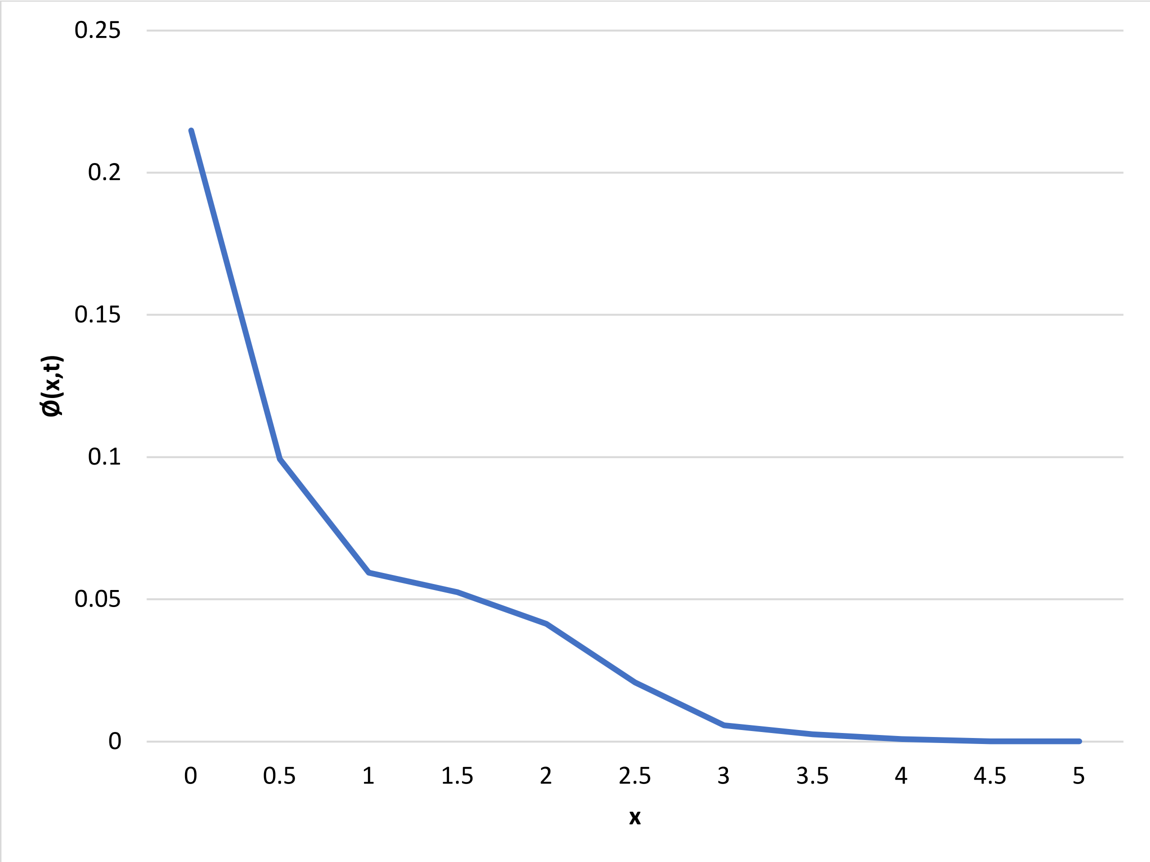

Fig. 5. Graph of exact and FEM solutions for equation (44)

By combing the above three graphs we can get approximate similarity solutions of equation (44)

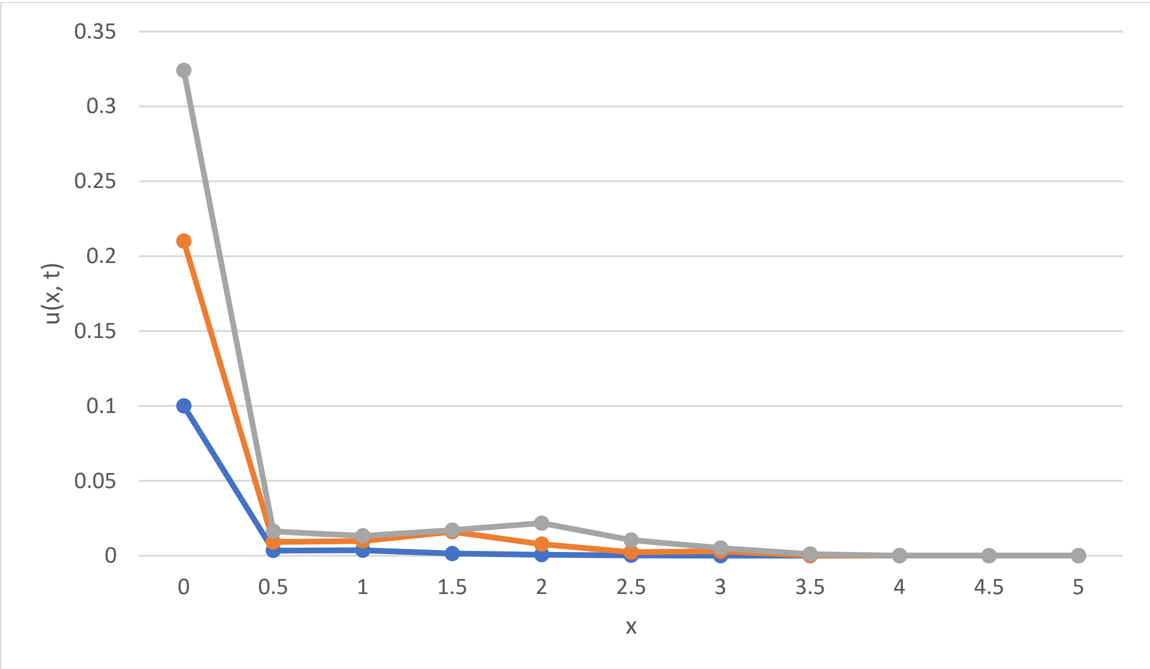

Fig. 6. Similarity between exact and approximate solutions of equation (44)

4 Results and Discussion

Table 1 makes it clear that the suggested strategy provides improved accuracy across the domain for various time periods. The features of the exact solution and the approximation have an interesting relationship. By applying the proven methodology of our proposed method, Figs. 1 and 2 can be generated from Eq. 44. The FEM shows how two graphs of approximate and exact results for various time steps agree. The mistake term can easily delete while still guaranteeing the validity and acceptance of our suggested method. The inaccuracy is, nevertheless, inherently plausible for comparably greater time steps.

Let's build up the three-dimensional surface plot of the numerical solution to Eq. 44 for easier comprehension. It's difficult to tell apart from this kind of comparison. This is the reason the absolute error map over time t and space x are also included. In the end, it makes sense that this approach is better suited to solving a CD equation of this kind without any complexity and maintaining perfect agreement between this FEM solution and the exact answer for equation 46, with a somewhat imitable error that tends to zero for tiny time steps. This test indicates that, in comparison to previous numerical techniques, the introduced FEM represents rapid convergence.

5 Conclusion

From the above study one can understand the analytical solution to Diffusion and convection partial differential equation of GCR, the energy equation of Galactic Cosmic Rays (GCR) and by combining all the graphs we can get an idea that there must be rapid convergence in FEM than other numerical methods.

Competing Interests

Authors have declared that no competing interests exist.

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