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Super Convergence Method for Solution of Higher Order Boundary Value Problems

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

A new $O(h^{10})$ super convergence method based on B-spline of degree eight has been developed for solution of higher order boundary value problems. Our presented collocation method leads to optimal approximation, we describe the mathematical procedure in detail also analyze the convergence of the method. The obtained numerical results have been compared with results obtained by recent existing methods to verify the applicability and super convergence properties of the presented method numerically.

Keywords: Boundary value problem; B-spline collocation method; super convergence; Green's function.

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1 Introduction

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, astronomy, beam and long wave theory, quantum mechanics, induction motors, engineering and applied physics. Many

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researchers have attempt to developed numerical solution of higher order boundary value problems. Many mathematical models arising in various applications can be written as boundary value problems, eighth and seventh order boundary value problems arise in the mathematical modeling of the fluid dynamics and induction motors respectively [1,2]. The solution of eighth order BVPs by differential quadrature rule is given by Liu and Wu [3] without any convergence analysis. Adomian decomposition and homotopy perturbation method have been given by [4,5] without any convergence analysis. Non-polynomial spline technique proposed by Siddiqi and Akram [6], but they obtained second order convergence. Reproducing kernel space method proposed by Akram and Rehman [7,8]. Variational iteration and differential transformation method are given by Siddiqi et al [9,10]. Many researchers applied collocation methods for solution of BVPs [11,12,13]. The cubic spline has been used to solve BVPs by pioneers [14,15,16,17], but their methods have second order convergent. An $O(h^4)$ optimal cubic spline collocation method was developed by Danial and Swatrz in [18]. Another optimal collocation method on mid-points based on quadratic spline to approximate the second order BVPs was proposed by Houstis et al. [19]. Irodotou-Ellina and Houstis in [20] developed an $O(h^6)$ optimal collocation method based on quintic spline for solving linear fourth order two point BVPs. In [21] Rashidinia and Ghasemi developed an optimal method based on sextic spline at the grid points for solving of nonlinear fifth order two point BVPs, after that they developed collocation method based on B-spline at the mid-points for the numerical solution of nonlinear sixth order BVPs [22]. The linear dependence relations for polynomial splines and error bounds for interpolating spline have been presented in [23,24].

In the present work, we will focus on developing a collocation method based on B-spline of degree eight to approximate the solution of the following nonlinear two point boundary value problems:

$$L_p u \equiv u^{(p)}(x) - \phi(x, u(x), u'(x), ..., u^{(p-1)}(x)) = 0, \qquad 1 \le p \le 8, a \le x \le b, \tag{1.1}$$

with the boundary conditions,

$$B_p u \equiv \sum_{j=0}^{p-1} (\alpha_{ij} u^{(j)}(a) + \beta_{ij} u^{(j)}(b)) = \gamma_i, 0 \le i \le p-1, \tag{1.2}$$

where α_{ij}, β_{ij} and γ_i are given real constants, ϕ is a continuous function, $u(x)$ is a unknown function, and L_p and B_p are differential operators. In this paper we will derive super convergence approximations of order $O(h^{10})$ at mid-points and grid points of the partition Δ on the interval $[a,b]$. In section 2, we obtain the consistency relations for spline of degree eight at mid-points and grid points of the partition Δ to construct higher order approximation. Section 3, is devoted to deriving the new super convergence collocation method based on spline of degree eight. The convergence analysis of the presented method is given in detail, in section 4. In section 5 numerical experiments are given to demonstrate the efficiency of the proposed method, we compared our numerical results with the results reported in [5-8,22,25-28]. The paper ends with conclusion.

2 Spline Interpolation

In this section we define spline interpolant $S(x)$ of degree eight that satisfies certain end conditions and then derive several relations that are required in the formulation of the collocation method. Now let $\Delta \equiv \{a = x_0 < x_1 < ... < x_n = b\}$ be a uniform partition of the interval $[a,b]$ with step size $h = \frac{b-a}{n}$.

Suppose that τ is the set of mid-points of the partition Δ include the boundary points as follows:

$$\tau \equiv \{t_0 = x_0, t_1 = \frac{x_0 + x_1}{2}, ..., t_i = \frac{x_{i-1} + x_i}{2}, ..., t_n = \frac{x_{n-1} + x_n}{2}, t_{n+1} = x_n\}$$

Consider a smooth spline of degree eight $S(x)$ that is an element of $SP_8(\Delta) \equiv \{\upsilon(x) \mid \upsilon(x) \in [a,b]\}$, and $\upsilon(x)$ is a polynomial of degree at most 8 on the partition Δ. By making use of a B-spline function

$$\mathbf{B}_{k+1}(x) = \frac{1}{k!}\sum_{j=0}^{k+1}(-1)^j \binom{k+1}{j}(x-j)_+^k \;, (x-j)_+^k = \begin{cases}(x-j)^k & x > j,\\ 0 & x < j,\end{cases}, S(x) = \sum_{j=-3}^{n+4} c_j \mathbf{B}_{k+1}\left(\frac{x-a}{h} - j + 5\right)$$

we consider a polynomial spline of degree eight $S(x)$ of the above form that satisfies the following interpolatory conditions:

$$S(t_i) = u(t_i), \quad 1 \le i \le n, \tag{2.1}$$

associated with the end conditions:

$$S^{(8)}(t_i) = u^{(8)}(t_i) - \frac{h^2}{24}u^{(10)}(t_i) + \frac{7h^4}{5760}u^{(12)}(t_i) - \frac{31h^6}{967680}u^{(14)}(t_i) + \frac{127h^8}{154828800}u^{(16)}(t_i), \tag{2.2}$$
$$i = 1, 2, 3, 4, n-3, n-2, n-1, n.$$

In the following, for sake of convenience we will denote that, $u_i^j \equiv u^j(t_i)$ and $S_i^j \equiv S^j(t_i)$, $i = 0, ..., n+1$, $0 \le j \le 8$ where $f^{(j)} \equiv D^{(j)} f$. By using [23] we have the following consistency relations for spline of degree eight S , at the mid-points for $5 \le i \le n-4$:

$$\Upsilon S_i^{(8)} = \frac{10321920}{h^8}(S_{i\pm4} - 8S_{i\pm3} + 28S_{i\pm2} - 56S_{i\pm1} + 70S_i), (a)$$
$$\Upsilon S_i^{(7)} = \frac{5160960}{h^7}(\mp S_{i\pm4} \pm 6S_{i\pm3} \mp 14S_{i\pm2} \pm 14S_{i\pm1}), (b)$$
$$\Upsilon S_i^{(6)} = \frac{1290240}{h^6}(S_{i\pm4} - 20S_{i\pm2} + 64S_{i\pm1} - 90S_i), (c)$$
$$\Upsilon S_i^{(5)} = \frac{215040}{h^5}(\mp S_{i\pm4} \mp 18S_{i\pm3} \pm 82S_{i\pm2} \mp 106S_{i\pm1}), (d)$$
$$\Upsilon S_i^{(4)} = \frac{26880}{h^4}(S_{i\pm4} + 72S_{i\pm3} - 68S_{i\pm2} - 392S_{i\pm1} + 774S_i), (e)$$
$$\Upsilon S_i^{(3)} = \frac{2688}{h^3}(\mp S_{i\pm4} \mp 234S_{i\pm3} \mp 974S_{i\pm2} \pm 2654S_{i\pm1}), (f)$$
$$\Upsilon S_i^{(2)} = \frac{224}{h^2}(S_{i\pm4} + 720S_{i\pm3} + 9100S_{i\pm2} + 3184S_{i\pm1} - 26010S_i), (g)$$
$$\Upsilon S_i^{(1)} = \frac{16}{h}(\mp S_{i\pm4} \mp 2178S_{i\pm3} \mp 58478S_{i\pm2} \mp 199066S_{i\pm1}), (h)$$
$$\Upsilon f_i \equiv f_{i\pm4} + 6552f_{i\pm3} + 331612f_{i\pm2} + 2485288f_{i\pm1} + 4675014f_i, (i) \tag{2.3}$$

where the discrete operator Υ is defined for any function f on the interval $[a,b]$. Now we will prove the following theorem to obtain the error bounds for spline $S(x)$ of degree eight and its derivatives up to order eight at the mid-points t_i, $i=1,...,n$ of the partition Δ.

Theorem 2.1: Let $S(x)$ be the unique spline of degree eight satisfying (2.1)-(2.2) and interpolating the function $u \in C^{18}[a,b]$, then for $i=1,...,n$ the following relations hold:

$$
\begin{aligned}
S_i^{(1)} &= u_i^{(1)} - \frac{127h^8}{154828800}u_i^{(9)} + O(h^{10}), (a)\\
S_i^{(2)} &= u_i^{(2)} + \frac{127h^8}{22118400}u_i^{(10)} + O(h^{10}), (b)\\
S_i^{(3)} &= u_i^{(3)} + \frac{31h^6}{967680}u_i^{(9)} - \frac{127h^8}{5529600}u_i^{(11)} + O(h^{10}), (c)\\
S_i^{(4)} &= u_i^{(4)} - \frac{31h^6}{193536}u_i^{(10)} + \frac{127h^8}{4423680}u_i^{(12)} + O(h^{10}), (d)\\
S_i^{(5)} &= u_i^{(5)} - \frac{7h^4}{5760}u_i^{(9)} + \frac{31h^6}{96768}u_i^{(11)} - \frac{127h^8}{4423680}u_i^{(13)} + O(h^{10}), (e)\\
S_i^{(6)} &= u_i^{(6)} + \frac{7h^4}{1920}u_i^{(10)} - \frac{31h^6}{96768}u_i^{(12)} + \frac{127h^8}{7372800}u_i^{(14)} + O(h^{10}), (f)\\
S_i^{(7)} &= u_i^{(7)} + \frac{h^2}{24}u_i^{(9)} - \frac{7h^4}{1920}u_i^{(11)} + \frac{31h^6}{193536}u_i^{(13)} - \frac{127h^8}{22118400}u_i^{(15)} + O(h^{10}), (g)\\
S_i^{(8)} &= u_i^{(8)} - \frac{h^2}{24}u_i^{(10)} + \frac{7h^4}{5760}u_i^{(12)} - \frac{31h^6}{967680}u_i^{(14)} + \frac{127h^8}{154828800}u_i^{(16)} + O(h^{10}), (h)
\end{aligned}
\tag{2.4}
$$

and further satisfying the following error bounds:

$$|(S-u)^{(k)}| = O(h^{9-k}), \quad k=0,...,7. \tag{2.5}$$

The above relations are held at the mid-points of the partition Δ too.

Proof. We will prove the relation (2.4)h, then by using this relation we can prove the rest of relations (2.4). Using Taylor's series expansion and taking into account that $S_i = u_i$, $i=1,...,n$, in the relation (2.3)a we obtain,

$$\Upsilon S_i^{(8)} = 10321920u_i^{(8)} + 3440640h^2u_i^{(10)} + 544768h^4u_i^{(12)} + \frac{163840}{3}h^6u_i^{(14)} + \frac{58496}{15}h^8u_i^{(16)} + O(h^{10}), \tag{2.6}$$

Further, using Taylor's series expansion for any function $f \in C^{18}[a,b]$, we conclude that

$$\Upsilon f_i == 10321920 f_i + 3870720 h^2 f_i^{(2)} + 69350 h^4 f_i^{(4)} + 79136 h^6 f_i^{(6)} + \frac{32349}{5} h^8 f_i^{(8)} + O(h^{10}), \qquad (2.7)$$

Setting $f(x) = u_i^{(8)} - \frac{h^2}{24} u_i^{(10)} + \frac{7h^4}{5760} u_i^{(12)} - \frac{31h^6}{967680} u_i^{(14)} + \frac{127h^8}{154828800} u_i^{(16)}$, we can obtain,

$$\Upsilon f_i = \Upsilon\left(u_i^{(8)} - \frac{h^2}{24} u_i^{(10)} + \frac{7h^4}{5760} u_i^{(12)} - \frac{31h^6}{967680} u_i^{(14)} + \frac{127h^8}{154828800} u_i^{(16)}\right) = 10321920 u_i^{(8)} + \\ 3440640 h^2 u_i^{(10)} + 544768 h^4 u_i^{(12)} + \frac{163840}{3} h^6 u_i^{(14)} + \frac{58496}{15} h^8 u_i^{(16)} + O(h^{10}), 5 \le i \le n-4. \qquad (2.8)$$

By subtracting equation (2.6) from (2.8) we obtain:

$$\Upsilon\left(S_i^{(8)} - u_i^{(8)} + \frac{h^2}{24} u_i^{(10)} - \frac{7h^4}{5760} u_i^{(12)} + \frac{31h^6}{967680} u_i^{(14)} - \frac{127h^8}{154828800} u_i^{(16)}\right) = O(h^{10}) \qquad (2.9)$$

Denoting $d_i \equiv S_i^{(8)} - u_i^{(8)} + \frac{h^2}{24} u_i^{(10)} - \frac{7h^4}{5760} u_i^{(12)} + \frac{31h^6}{967680} u_i^{(14)} - \frac{127h^8}{154828800} u_i^{(16)}$, then by using the end conditions (2.2) and consistency equation (2.9) we obtain,

$$\Upsilon d_i = O(h^{10} |u^{(18)}|), \ 5 \le i \le n-4, \qquad d_1 = d_2 = d_3 = d_4 = d_{n-3} = d_{n-2} = d_{n-1} = d_n = 0. \qquad (2.10)$$

Since the coefficient matrix of the above system is positive definite hence, it is nonsingular, and its inverse has a finite norm, thus we have $d_i = O(h^{10}), i = 1,...,n$, so the proof of relation (2.4)h is completed. To prove relation (2.4)g we consider the following consistency relations, which can be easily obtained (but long straightforward calculations) for any spline of degree eight at the interior mid-points t_i,

$$S_i^{(7)} = \frac{1}{10321920 h^7}[h^8(S_i^{(8)} - 5154407 S_{i+1}^{(8)} - 9983755 S_{i+2}^{(8)} - 7498467 S_{i+3}^{(8)} - 2823453 S_{i+4}^{(8)} - \\ 338165 S_{i+5}^{(8)} - 6553 S_{i+6}^{(8)} - S_{i+7}^{(8)}) \mp 10321920 S_{i,i+7} \pm 72253440 S_{i+1,i+6} \mp 216760320 S_{i+2,i+5} \pm \\ 361267200 S_{i+3,i+4}], 1 \le i \le n-7,$$

$$S_i^{(7)} = \frac{1}{10321920 h^7}[h^8(S_{i-7}^{(8)} + 6553 S_{i-6}^{(8)} + 338165 S_{i-5}^{(8)} + 2823453 S_{i-4}^{(8)} + 7498467 S_{i-3}^{(8)} + 9983755 S_{i-2}^{(8)} + \\ 10315367 S_{i-1}^{(8)} + 5160959 S_i^{(8)}) \pm 10321920 S_{i,i-7} \mp 72253440 S_{i-1,i-6} \pm 216760320 S_{i-2,i-5} \mp \\ 361267200 S_{i-3,i-4}], 8 \le i \le n.$$

Using relation (2.4)h in the above relation and applying Taylor's series expansions of $u_{i\pm l}^{(k)}$ for $k = 0,10,12,14,16$, we can obtain,

$$S_i^{(7)} = u_i^{(7)} + \frac{h^2}{24} u_i^{(9)} - \frac{7h^4}{1920} u_i^{(11)} + \frac{31h^6}{193536} u_i^{(13)} - \frac{127h^8}{22118400} u_i^{(15)} + O(h^{10}), 1 \le i \le n.$$

In a similar manner by using any appropriate consistency relations we can prove the other relations in this Theorem. According to the relation (2.5), it is well known that if the spline of degree eight has been applied for solving interpolating problems and boundary value problems of order $k, k=0,...,7$ has at most $O(h^9)$ and $O(h^{9-k})$ accuracy, respectively.

Theorem 2.2: Let $S(x)$ be the unique spline of degree eight satisfying (2.1)-(2.2) and interpolating the function $u \in C^{18}[a,b]$, then for $i=1,...,n$ the following relations hold:

$$S^{(1)}(x_i)=u^{(1)}(x_i)+\frac{h^8}{1209600}u^{(9)}(x_i)+O(h^{10}), (a)$$
$$S^{(2)}(x_i)=u^{(2)}(x_i)-\frac{h^8}{172800}u^{(10)}(x_i)+O(h^{10}), (b)$$
$$S^{(3)}(x_i)=u^{(3)}(x_i)-\frac{h^6}{30240}u^{(9)}(x_i)+\frac{h^8}{57600}u^{(11)}(x_i)+O(h^{10}), (c)$$
$$S^{(4)}(x_i)=u^{(4)}(x_i)+\frac{h^6}{6048}u^{(10)}(x_i)-\frac{h^8}{34560}u^{(12)}(x_i)+O(h^{10}), (d)$$
$$S^{(5)}(x_i)=u^{(5)}(x_i)+\frac{h^4}{720}u^{(9)}(x_i)-\frac{h^6}{3024}u^{(11)}(x_i)+\frac{h^8}{34560}u^{(13)}(x_i)+O(h^{10}), (e)$$
$$S^{(6)}(x_i)=u^{(6)}(x_i)-\frac{h^4}{240}u^{(10)}(x_i)+\frac{h^6}{3024}u^{(12)}(x_i)-\frac{h^8}{57600}u^{(14)}(x_i)+O(h^{10}), (f)$$
$$S^{(7)}(x_i)=u^{(7)}(x_i)-\frac{h^2}{12}u^{(9)}(x_i)+\frac{h^4}{240}u^{(11)}(x_i)-\frac{h^6}{6048}u^{(13)}(x_i)+\frac{h^8}{172800}u^{(15)}(x_i)+O(h^{10}). (g)$$
(2.11)

The above relations are held at the grid points of the partition Δ too.

Proof. The proof is similar to the proof of theorem 2.1.

To derive a super convergence method for the solution of equations (1.1) and (1.2) using Theorems 2.1 and 2.2, we need to define some appropriate relations to connect $u^{(9)},...,u^{(16)}$ with spline $S(x)$ and its first eighth derivatives $S',...,S^{(8)}$ at the mid-points as well as at grid points of the partition Δ. For sake of convenience, we define the following discrete operators for $5 \le i \le n-4$:

$$\mu_1 g_i = g_{i\pm4}-8g_{i\pm3}+28g_{i\pm2}-56g_{i\pm1}+70g_i,$$
$$\mu_2 g_i = \frac{1}{4}(-g_{i\pm4}+12g_{i\pm3}-52g_{i\pm2}+116g_{i\pm1}-150g_i),$$
$$\mu_3 g_i = \frac{1}{24}(-7g_{i\pm4}+80g_{i\pm3}-340g_{i\pm2}+752g_{i\pm1}-970g_i),$$
$$\mu_4 g_i = \frac{1}{24}(-5g_{i\pm4}+64g_{i\pm3}-284g_{i\pm2}+640g_{i\pm1}-830g_i),$$
$$\mu_5 g_i = \frac{1}{1152}(32g_{i\pm4}-472g_{i\pm3}+328g_{i\pm2}-9448g_{i\pm1}+13202g_i),$$

$$\mu_6 g_i = \frac{1}{1920}(49g_{i\pm4} - 712g_{i\pm3} + 5212g_{i\pm2} - 15224g_{i-1} + 21350g_i),$$
$$\mu_7 g_i = \frac{1}{5760}(259g_{i\pm4} - 3272g_{i\pm3} + 20212g_{i\pm2} - 55544g_{i\pm1} + 76690g_i),$$
$$\mu_8 g_i = \frac{1}{1920}(37g_{i\pm4} - 536g_{i\pm3} + 4396g_{i\pm2} - 13352g_{i\pm1} + 18910g_i), \tag{2.12}$$
$$\mu_9 g_i = \frac{1}{1024}(-5g_{i\pm4} + 64g_{i\pm3} - 412g_{i\pm2} + 2176g_{i\pm1} - 3646g_i),$$
$$\mu_{10} g_i = \frac{1}{107520}(-75g_{i\pm4} + 1104g_{i\pm3} - 9604g_{i\pm2} + 137200g_{i\pm1} - 257250g_i),$$

Lemma 2.1: If $u \in C^{18}[a,b]$ and $5 \le i \le n-4$, then using the above operators we have

$$u_i^{(r)} = \frac{\mu_1 S_i^{(r-8)}}{h^8} + O(h^2),\ 9 \le r \le 16,\ u_i^{(r)} = \frac{\mu_2 S_i^{(r-8)}}{h^6} + O(h^4),\ 9 \le r \le 12,$$
$$u_i^{(13)} = \frac{\mu_3 S_i^{(7)}}{h^6} + O(h^4), u_i^{(14)} = \frac{\mu_4 S_i^{(8)}}{h^6} + O(h^4), u_i^{(9)} = \frac{\mu_5 S_i^{(5)}}{h^4} + O(h^6),$$
$$u_i^{(10)} = \frac{\mu_6 S_i^{(6)}}{h^4} + O(h^6), u_i^{(11)} = \frac{\mu_7 S_i^{(7)}}{h^4} + O(h^6), u_i^{(12)} = \frac{\mu_8 S_i^{(8)}}{h^4} + O(h^6),$$
$$u_i^{(9)} = \frac{\mu_9 S_i^{(7)}}{h^2} + O(h^8), u_i^{(10)} = \frac{\mu_{10} S_i^{(8)}}{h^2} + O(h^8).$$

Proof. Using relation (2.12) and the results of Theorem 2.1, these relations can be proved easily.

Corollary 2.1: If $S(x)$ be the unique spline of degree eight interpolating $u \in C^{18}[a,b]$, then for $5 \le i \le n-4$ the following relations hold:

$$u_i^{(8)} = S_i^{(8)} + \frac{1}{24}\mu_{10} S_i^{(8)} - \frac{7}{5760}\mu_8 S_i^{(8)} + \frac{31}{967680}\mu_4 S_i^{(8)} - \frac{127}{154828800}\mu_1 S_i^{(8)} + O(h^{10}),$$
$$u_i^{(7)} = S_i^{(7)} - \frac{1}{24}\mu_9 S_i^{(7)} + \frac{7}{1920}\mu_7 S_i^{(7)} - \frac{31}{193536}\mu_3 S_i^{(7)} + \frac{127}{22118400}\mu_1 S_i^{(7)} + O(h^{10}),$$
$$u_i^{(6)} = S_i^{(6)} - \frac{7}{1920}\mu_6 S_i^{(6)} + \frac{31}{96768}\mu_2 S_i^{(6)} - \frac{127}{7372800}\mu_1 S_i^{(6)} + O(h^{10}),$$
$$u_i^{(5)} = S_i^{(5)} + \frac{7}{5760}\mu_5 S_i^{(5)} - \frac{31}{96768}\mu_2 S_i^{(5)} + \frac{127}{4423680}\mu_1 S_i^{(5)} + O(h^{10}),$$
$$u_i^{(4)} = S_i^{(4)} + \frac{31}{193536}\mu_2 S_i^{(4)} - \frac{127}{4423680}\mu_1 S_i^{(4)} + O(h^{10}),$$
$$u_i^{(3)} = S_i^{(3)} - \frac{31}{967680}\mu_2 S_i^{(3)} + \frac{127}{5529600}\mu_1 S_i^{(3)} + O(h^{10}),$$

$$u_i^{(2)} = S_i^{(2)} - \frac{127}{22118400}\mu_1 S_i^{(2)} + O(h^{10}),$$
$$u_i^{(1)} = S_i^{(1)} + \frac{127}{154828800}\mu_1 S_i^{(1)} + O(h^{10}).$$

Remark: The Corollary 2.1 gives the approximation for $u^{(p)}, 1 \le p \le 8$ at the interior mid-points of the partition Δ which is the conclusion of Lemma 2.1 and Theorem 2.1. Now we need to obtain some similar relations at the boundaries and its neighboring points. In order to obtain approximations for derivatives at the boundaries and its neighboring points $\{x_0, t_1, t_2, t_3, t_4, t_{n-3}, t_{n-2}, t_{n-1}, t_n, x_n\}$ we should use Taylor's series expansions.

Lemma 2.2: If $u \in C^{18}[a,b]$ and $\sigma_i = i, i = 0,1,2,3,4$ denoting the grid points, near the left end point x_0 and $\sigma_i = n-i, i = n-3, n-2, n-1, n, n+1$ denoting the grid points, near the right end point x_n, that σ_i is the index of τ, then the following $O(h^2)$ approximations to the higher order derivatives of u hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{1}{2h^8}\mu_1\left(23S_{\sigma_5}^{(r-8)} - 19S_{\sigma_6}^{(r-8)}\right) + O(h^2), u_{\sigma_1}^{(r)} = \frac{1}{h^8}\mu_1\left(5S_{\sigma_5}^{(r-8)} - 4S_{\sigma_6}^{(r-8)}\right) + O(h^2),$$
$$u_{\sigma_2}^{(r)} = \frac{1}{h^8}\mu_1\left(4S_{\sigma_5}^{(r-8)} - 3S_{\sigma_6}^{(r-8)}\right) + O(h^2), u_{\sigma_3}^{(r)} = \frac{1}{h^8}\mu_1\left(3S_{\sigma_5}^{(r-8)} - 2S_{\sigma_6}^{(r-8)}\right) + O(h^2),$$
$$u_{\sigma_4}^{(r)} = \frac{1}{h^8}\mu_1\left(2S_{\sigma_5}^{(r-8)} - S_{\sigma_6}^{(r-8)}\right) + O(h^2),\ 9 \le r \le 16.$$

Proof. To prove Lemma 2.2, we consider the following relations, which can be easily obtained by finite difference,

$$u_{\sigma_0}^{(r)} = \frac{1}{2}(23u_{\sigma_5}^{(r)} - 19u_{\sigma_6}^{(r)}) + O(h^2), u_{\sigma_1}^{(r)} = 5u_{\sigma_5}^{(r)} - 4u_{\sigma_6}^{(r)} + O(h^2),$$
$$u_{\sigma_2}^{(r)} = 4u_{\sigma_5}^{(r)} - 3u_{\sigma_6}^{(r)} + O(h^2), u_{\sigma_3}^{(r)} = 3u_{\sigma_5}^{(r)} - 2u_{\sigma_6}^{(r)} + O(h^2),$$
$$u_{\sigma_4}^{(r)} = 2u_{\sigma_5}^{(r)} - u_{\sigma_6}^{(r)} + O(h^2),\ 9 \le r \le 16.$$

By using the results of Lemma 2.1 and above relations, the proof is completed.

Lemma 2.3: Under the assumptions of Lemma 2.2, we have the following $O(h^4)$ approximations to the higher order derivatives of u hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{1}{16h^6}\mu_2(715S_{\sigma_5}^{(r-6)} - 1755S_{\sigma_6}^{(r-6)} + 1485S_{\sigma_7}^{(r-6)} - 429S_{\sigma_8}^{(r-6)}) + O(h^4),$$
$$u_{\sigma_1}^{(r)} = \frac{1}{h^6}\mu_2(35S_{\sigma_5}^{(r-6)} - 84S_{\sigma_6}^{(r-6)} + 70S_{\sigma_7}^{(r-6)} - 20S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^6}\mu_2(20S_{\sigma_5}^{(r-6)} - 45S_{\sigma_6}^{(r-6)} + 36S_{\sigma_7}^{(r-6)} - 10S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_3}^{(r)} = \frac{1}{h^6}\mu_2(10S_{\sigma_5}^{(r-6)} - 20S_{\sigma_6}^{(r-6)} + 15S_{\sigma_7}^{(r-6)} - 4S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_4}^{(r)} = \frac{1}{h^6}\mu_2(4S_{\sigma_5}^{(r-6)} - 6S_{\sigma_6}^{(r-6)} + 4S_{\sigma_7}^{(r-6)} - S_{\sigma_8}^{(r-6)}) + O(h^4),\ 9 \le r \le 12,$$

$$u_{\sigma_0}^{(13)} = \frac{1}{16h^6}\mu_3(715S_{\sigma_5}^{(7)} - 1755S_{\sigma_6}^{(7)} + 1485S_{\sigma_7}^{(7)} - 429S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_1}^{(13)} = \frac{1}{h^6}\mu_3(35S_{\sigma_5}^{(7)} - 84S_{\sigma_6}^{(7)} + 70S_{\sigma_7}^{(7)} - 20S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_2}^{(13)} = \frac{1}{h^6}\mu_3(20S_{\sigma_5}^{(7)} - 45S_{\sigma_6}^{(7)} + 36S_{\sigma_7}^{(7)} - 10S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_3}^{(13)} = \frac{1}{h^6}\mu_3(10S_{\sigma_5}^{(7)} - 20S_{\sigma_6}^{(7)} + 15S_{\sigma_7}^{(7)} - 4S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_4}^{(13)} = \frac{1}{h^6}\mu_3(4S_{\sigma_5}^{(7)} - 6S_{\sigma_6}^{(7)} + 4S_{\sigma_7}^{(7)} - S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_0}^{(14)} = \frac{1}{16h^6}\mu_4(715S_{\sigma_5}^{(8)} - 1755S_{\sigma_6}^{(8)} + 1485S_{\sigma_7}^{(8)} - 429S_{\sigma_8}^{(8)}) + O(h^4),$$

$$u_{\sigma_1}^{(14)} = \frac{1}{h^6}\mu_4(35S_{\sigma_5}^{(8)} - 84S_{\sigma_6}^{(8)} + 70S_{\sigma_7}^{(8)} - 20S_{\sigma_8}^{(8)}) + O(h^4)$$

$$u_{\sigma_2}^{(14)} = \frac{1}{h^6}\mu_4(20S_{\sigma_5}^{(8)} - 45S_{\sigma_6}^{(8)} + 36S_{\sigma_7}^{(8)} - 10S_{\sigma_8}^{(8)}) + O(h^4),$$

$$u_{\sigma_3}^{(14)} = \frac{1}{h^6}\mu_4(10S_{\sigma_5}^{(8)} - 20S_{\sigma_6}^{(8)} + 15S_{\sigma_7}^{(8)} - 4S_{\sigma_8}^{(8)}) + O(h^4),$$

$$u_{\sigma_4}^{(14)} = \frac{1}{h^6}\mu_4(4S_{\sigma_5}^{(8)} - 6S_{\sigma_6}^{(8)} + 4S_{\sigma_7}^{(8)} - S_{\sigma_8}^{(8)})) + O(h^4).$$

Proof. The proof is similar to the proof of Lemma 2.2.

Lemma 2.4: Under the assumptions of Lemma 2.2, we have the following $O(h^6)$ approximations hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{1}{256h^4}\mu_l(46189S_{\sigma_5}^{(l)} - 188955S_{\sigma_6}^{(l)} + 159885S_{\sigma_7}^{(l)} - 138567S_{\sigma_8}^{(l)} + 122265S_{\sigma_9}^{(l)} - 21879S_{\sigma_{10}}^{(l)}) +$$
$$O(h^6), r = 9,10,\ l = 5,6$$

$$u_{\sigma_1}^{(r)} = \frac{1}{h^4}\mu_l(126S_{\sigma_5}^{(l)} - 504S_{\sigma_6}^{(l)} + 840S_{\sigma_7}^{(l)} - 720S_{\sigma_8}^{(l)} + 315S_{\sigma_9}^{(l)} - 56S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9,10,\ l = 5,6$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^4}\mu_l(56S_{\sigma_5}^{(l)} - 210S_{\sigma_6}^{(l)} + 336S_{\sigma_7}^{(l)} - 280S_{\sigma_8}^{(l)} + 120S_{\sigma_9}^{(l)} - 21S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9,10,\ l = 5,6$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^4}\mu_l(56S_{\sigma_5}^{(l)} - 210S_{\sigma_6}^{(l)} + 336S_{\sigma_7}^{(l)} - 280S_{\sigma_8}^{(l)} + 120S_{\sigma_9}^{(l)} - 21S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9,10,\ l = 5,6$$

$$u_{\sigma_3}^{(r)} = \frac{1}{h^4}\mu_l(21S_{\sigma_5}^{(l)} - 70S_{\sigma_6}^{(l)} + 105S_{\sigma_7}^{(l)} - 84S_{\sigma_8}^{(l)} + 35S_{\sigma_9}^{(l)} - 6S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9,10,\ l = 5,6$$

$$u_{\sigma_4}^{(r)} = \frac{1}{h^4}\mu_l(6S_{\sigma_5}^{(l)} - 15S_{\sigma_6}^{(l)} + 20S_{\sigma_7}^{(l)} - 15S_{\sigma_8}^{(l)} + 6S_{\sigma_9}^{(l)} - S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9,10,\ l = 5,6$$

$$u_{\sigma_0}^{(r)} = \frac{1}{256h^4}\mu_l(46189S_{\sigma_5}^{(l)} - 188955S_{\sigma_6}^{(l)} + 159885S_{\sigma_7}^{(l)} - 138567S_{\sigma_8}^{(l)} + 122265S_{\sigma_9}^{(l)}$$
$$-21879S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11,12,\ l = 7,8,$$

$$u_{\sigma_1}^{(r)} = \frac{1}{h^4}\mu_l(126S_{\sigma_5}^{(l)} - 504S_{\sigma_6}^{(l)} + 840S_{\sigma_l}^{(l)} - 720S_{\sigma_8}^{(l)} + 122265S_{\sigma_9}^{(l)} - 21879S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11,12,\ l = 7,8,$$

$$u_{\sigma_1}^{(r)} = \frac{1}{h^4}\mu_l(126S_{\sigma_5}^{(l)} - 504S_{\sigma_6}^{(l)} + 840S_{\sigma_l}^{(l)} - 720S_{\sigma_8}^{(l)} + 315S_{\sigma_9}^{(l)} - 56S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11,12,\ l = 7,8,$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^4}\mu_l(56S_{\sigma_5}^{(l)} - 210S_{\sigma_6}^{(l)} + 336S_{\sigma_7}^{(l)} - 280S_{\sigma_8}^{(l)} + 120S_{\sigma_9}^{(l)} - 21S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11,12,\ l = 7,8,$$

$$u_{\sigma_3}^{(r)} = \frac{1}{h^4}\mu_l(21S_{\sigma_5}^{(l)} - 70S_{\sigma_6}^{(l)} + 105S_{\sigma_7}^{(l)} - 84S_{\sigma_8}^{(l)} + 35S_{\sigma_9}^{(l)} - 6S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11,12,\ l = 7,8,$$

$$u_{\sigma_4}^{(r)} = \frac{1}{h^4}\mu_l(6S_{\sigma_5}^{(l)} - 15S_{\sigma_6}^{(l)} + 20S_{\sigma_7}^{(l)} - 15S_{\sigma_8}^{(l)} + 6S_{\sigma_9}^{(l)} - S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11,12,\ l = 7,8.$$

Proof. The proof is similar to the proof of Lemma 2.2.

Lemma 2.5: Under the assumptions of Lemma 2.2, we have the following $O(h^8)$ approximations to the higher order derivatives of u hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{\mu_r}{2048h^2}(1062347S_{\sigma_5}^{(l)} - 6084351S_{\sigma_6}^{(l)} + 15444891S_{\sigma_7}^{(l)} - 22309287S_{\sigma_8}^{(l)} + 1984665S_{\sigma_9}^{(l)}$$
$$-10567557S_{\sigma_{10}}^{(l)} + 3187041S_{\sigma_{11}}^{(l)} - 415701S_{\sigma_{12}}^{(l)}) + O(h^8), r = 9,10,\ l = 7,8.$$

$$u_{\sigma_1}^{(r)} = \frac{\mu_r}{h^2}(330S_{\sigma_5}^{(l)} - 1848S_{\sigma_6}^{(l)} + 4620S_{\sigma_7}^{(l)} - 6600S_{\sigma_8}^{(l)} + 5775S_{\sigma_9}^{(l)} - 3080S_{\sigma_{10}}^{(l)} + 924S_{\sigma_{11}}^{(l)} - 120S_{\sigma_{12}}^{(l)})$$
$$+O(h^8), r = 9,10,\ l = 7,8.$$

$$u_{\sigma_2}^{(r)} = \frac{\mu_r}{h^2}(120S_{\sigma_5}^{(l)} - 630S_{\sigma_6}^{(l)} + 1512S_{\sigma_7}^{(l)} - 2100S_{\sigma_8}^{(l)} + 1800S_{\sigma_9}^{(l)} - 945S_{\sigma_{10}}^{(l)} + 280S_{\sigma_{11}}^{(l)} - 36S_{\sigma_{12}}^{(l)})$$
$$+O(h^8), r = 9,10,\ l = 7,8.$$

$$u_{\sigma_3}^{(r)} = \frac{\mu_r}{h^2}(36S_{\sigma_5}^{(l)} - 168S_{\sigma_6}^{(l)} + 378S_{\sigma_7}^{(l)} - 504S_{\sigma_8}^{(l)} + 420S_{\sigma_9}^{(l)} - 216S_{\sigma_{10}}^{(l)} + 634S_{\sigma_{11}}^{(l)} - 8S_{\sigma_{12}}^{(l)})$$
$$+O(h^8), r = 9,10,\ l = 7,8.$$

$$u_{\sigma_4}^{(r)} = \frac{\mu_r}{h^2}(8S_{\sigma_5}^{(l)} - 28S_{\sigma_6}^{(l)} + 56S_{\sigma_7}^{(l)} - 70S_{\sigma_8}^{(l)} + 56S_{\sigma_9}^{(l)} - 28S_{\sigma_{10}}^{(l)} + 8S_{\sigma_{11}}^{(l)} - S_{\sigma_{12}}^{(l)})$$
$$+O(h^8), r = 9,10,\ l = 7,8.$$

Proof. The proof is similar to the proof of Lemma 2.2.

3 Super Convergence Spline Collocation

3.1 Formulation of the method

Suppose that $u \in C^{10+p}[a,b]$, $1 \le p \le 8$ be the exact solution of the given BVP (1.1) and (1.2) and $S(x) \in Sp_8(\Delta)$ be the spline to approximate the solution of BVP. By discretization the BVP (1.1) and (1.2) at the points of τ we have:

$$\begin{aligned} L_p u_i &\equiv [u^{(p)}(x) - \phi(x, u(x), u'(x), \ldots, u^{(p-1)}(x))]_{t_i},\ 1 \le i \le n, \\ B_p u_i &\equiv \sum_{j=0}^{p-1} (\alpha_{ij} u^{(j)}(a) + \beta_{ij} u^{(j)}(b)) = \gamma_i,\ 0 \le i \le p-1, \end{aligned} \tag{3.1}$$

Replacing u by S and using relations in the Lemmas 2.2-2.5, we can obtain a system of $n+8$ nonlinear equations for each $1 \le p \le 8$. For example, hence we consider this system for the case of $p = 8$ and the other cases are similar,

$$\begin{aligned} L_{8,\sigma_1}^s S &= S_{\sigma_1}^{(8)} + \mu_{10}\xi^{(8)} - \mu_8\psi^{(8)} + \mu_4\chi^{(8)} - \eta^{(8)} - \phi(t_{\sigma_1}, S_{\sigma_1}, S'_{\sigma_1} + \eta^{(1)}, S''_{\sigma_1} - 7\eta^{(2)}, \\ & S_{\sigma_1}^{(3)} - \mu_2\chi^{(3)} + 28\eta^{(3)}, S_{\sigma_1}^{(4)} + 5\mu_2\chi^{(4)} - 35\eta^{(4)}, S_{\sigma_1}^{(5)} + \mu_5\psi^{(5)} - 10\mu_2\chi^{(5)} + 35\eta^{(5)}, \\ & S_{\sigma_1}^{(6)} - 3\mu_6\psi^{(6)} + 10\mu_2\chi^{(6)} - 21\eta^{(6)}, S_{\sigma_1}^{(7)} - \mu_9\xi^{(7)} + 3\mu_7\psi^{(7)} - 5\mu_3\chi^{(7)} + 7\eta^{(7)}) + O(h^{10}), i = 1, n, \end{aligned} \tag{3.2}$$

In the above system, just replacing ξ, ψ, χ and η by $\bar{\xi}, \bar{\psi}, \bar{\chi}$ and $\bar{\eta}$ we can obtain $L_{8,\sigma_2}^s S$ for $i = 2, n-1$ and by $\bar{\bar{\xi}}, \bar{\bar{\psi}}, \bar{\bar{\chi}}$ and $\bar{\bar{\eta}}$ we can obtain $L_{8,\sigma_3}^s S$ for $i = 3, n-2$. By $\tilde{\xi}, \tilde{\psi}, \tilde{\chi}$ and $\tilde{\eta}$ we can obtain $L_{8,\sigma_4}^s S$ for $i = 4, n-3$. By using relations of Corollary 2.1, we can obtain $L_{8,i}^s S$ for $5 \le i \le n-4$:

$$\begin{aligned} L_{8,i}^s S &= S_i^{(8)} + \frac{1}{24}\mu_{10}S_i^{(8)} - \frac{7}{5760}\mu_8 S_i^{(8)} + \frac{31}{967680}\mu_4 S_i^{(8)} - \frac{127}{154828800}\mu_1 S_i^{(8)} - \\ & \phi(t_i, S_i, S'_i + \frac{127}{154828800}\mu_1 S'_i, S''_i - \frac{127}{22118400}\mu_1 S''_i, S_i^{(3)} - \frac{31}{967680}\mu_2 S_i^{(3)} + \frac{127}{5529600}\mu_1 S_i^{(3)}, \\ & S_i^{(4)} + \frac{31}{193536}\mu_2 S_i^{(4)} - \frac{127}{4423680}\mu_1 S_i^{(4)}, S_i^{(5)} + \frac{7}{5760}\mu_5 S_i^{(5)} - \frac{31}{96768}\mu_2 S_i^{(5)} + \frac{127}{4423680}\mu_1 S_i^{(5)}, \\ & S_i^{(6)} - \frac{7}{1920}\mu_6 S_i^{(6)} + \frac{31}{96768}\mu_2 S_i^{(6)} - \frac{127}{7372800}\mu_1 S_i^{(6)}, S_i^{(7)} - \frac{1}{24}\mu_9 S_i^{(7)} + \frac{7}{1920}\mu_7 S_i^{(7)} - \\ & \frac{31}{193536}\mu_3 S_i^{(7)} + \frac{127}{22118400}\mu_1 S_i^{(7)}) + O(h^{10}), \qquad 5 \le i \le n-4, \end{aligned} \tag{3.3}$$

where

$$\xi^{(k)}=\frac{1}{24}(330S_{\sigma_5}^{(k)}-1848S_{\sigma_6}^{(k)}+4620S_{\sigma_7}^{(k)}-6600S_{\sigma_8}^{(k)}+5775S_{\sigma_9}^{(k)}-3080S_{\sigma_{10}}^{(k)}+924S_{\sigma_{11}}^{(k)}-120S_{\sigma_{12}}^{(k)}),k=7,8,$$

$$\bar{\xi}^{(k)}=\frac{1}{24}(120S_{\sigma_5}^{(k)}-630S_{\sigma_6}^{(k)}+1512S_{\sigma_7}^{(k)}-2100S_{\sigma_8}^{(k)}+1800S_{\sigma_9}^{(k)}-945S_{\sigma_{10}}^{(k)}+280S_{\sigma_{11}}^{(k)}-36S_{\sigma_{12}}^{(k)}),k=7,8,$$

$$\bar{\bar{\xi}}^{(k)}=\frac{1}{24}(36S_{\sigma_5}^{(k)}-168S_{\sigma_6}^{(k)}+378S_{\sigma_7}^{(k)}-504S_{\sigma_8}^{(k)}+420S_{\sigma_9}^{(k)}-216S_{\sigma_{10}}^{(k)}+63S_{\sigma_{11}}^{(k)}-8S_{\sigma_{12}}^{(k)}),k=7,8,$$

$$\tilde{\xi}^{(k)}=\frac{1}{24}(8S_{\sigma_5}^{(k)}-28S_{\sigma_6}^{(k)}+56S_{\sigma_7}^{(k)}-70S_{\sigma_8}^{(k)}+56S_{\sigma_9}^{(k)}-28S_{\sigma_{10}}^{(k)}+8S_{\sigma_{11}}^{(k)}-S_{\sigma_{12}}^{(k)}),k=7,8,$$

$$\psi^{(k)}=\frac{7}{5760}(126S_{\sigma_5}^{(k)}-504S_{\sigma_6}^{(k)}+840S_{\sigma_7}^{(k)}-720S_{\sigma_8}^{(k)}+315S_{\sigma_9}^{(k)}-56S_{\sigma_{10}}^{(k)}),k=5,...,8,$$

$$\bar{\psi}^{(k)}=\frac{7}{5760}(56S_{\sigma_5}^{(k)}-210S_{\sigma_6}^{(k)}+336S_{\sigma_7}^{(k)}-280S_{\sigma_8}^{(k)}+120S_{\sigma_9}^{(k)}-21S_{\sigma_{10}}^{(k)}),k=7,8,$$

$$\eta^{(k)}=\frac{127}{154828800}\mu_1(5S_{\sigma_5}^{(k)}-4S_{\sigma_6}^{(k)}),\bar{\eta}^{(k)}=\frac{127}{154828800}\mu_1(4S_{\sigma_5}^{(k)}-3S_{\sigma_6}^{(k)}),k=1,...,8,$$

$$\bar{\bar{\eta}}^{(k)}=\frac{127}{154828800}\mu_1(3S_{\sigma_5}^{(k)}-2S_{\sigma_6}^{(k)}),\tilde{\eta}^{(k)}=\frac{127}{154828800}\mu_1(2S_{\sigma_5}^{(k)}-S_{\sigma_6}^{(k)}),k=1,...,8,$$

$$\bar{\bar{\psi}}^{(k)}=\frac{7}{5760}(21S_{\sigma_5}^{(k)}-70S_{\sigma_6}^{(k)}+105S_{\sigma_7}^{(k)}-84S_{\sigma_8}^{(k)}+35S_{\sigma_9}^{(k)}-6S_{\sigma_{10}}^{(k)}),k=5,...,8,$$

$$\tilde{\psi}^{(k)}=\frac{7}{5760}(6S_{\sigma_5}^{(k)}-15S_{\sigma_6}^{(k)}+20S_{\sigma_7}^{(k)}-15S_{\sigma_8}^{(k)}+6S_{\sigma_9}^{(k)}-S_{\sigma_{10}}^{(k)}),k=5,...,8,$$

$$\chi^{(k)}=\frac{31}{967680}(35S_{\sigma_5}^{(k)}-84S_{\sigma_6}^{(k)}+70S_{\sigma_7}^{(k)}-20S_{\sigma_8}^{(k)}),k=3,...,8,$$

$$\bar{\chi}^{(k)}=\frac{31}{967680}(20S_{\sigma_5}^{(k)}-45S_{\sigma_6}^{(k)}+36S_{\sigma_7}^{(k)}-10S_{\sigma_8}^{(k)}),k=3,...,8,$$

$$\bar{\bar{\chi}}^{(k)}=\frac{31}{967680}(10S_{\sigma_5}^{(k)}-20S_{\sigma_6}^{(k)}+15S_{\sigma_7}^{(k)}-4S_{\sigma_8}^{(k)}),k=3,...,8,$$

$$\tilde{\chi}^{(k)}=\frac{31}{967680}(4S_{\sigma_5}^{(k)}-6S_{\sigma_6}^{(k)}+4S_{\sigma_7}^{(k)}-S_{\sigma_8}^{(k)}),k=3,...,8, \tag{3.4}$$

and the boundary formulas,

$$B_8^sS\equiv\alpha_{i,0}S_0+\beta_{i,0}S_{n+1}+\sum_{i,0}^{7}\alpha_{i,j}\theta_j+\sum_{i,0}^{7}\beta_{i,j}\bar{\theta}_j=\gamma_i+O(h^{10}),\qquad 0\le i\le 7, \tag{3.5}$$

With

$$\theta_k = \overline{\theta}_k = S_{\sigma_0}^{(k)} + \mu_k^* (23S_{\sigma_5}^{(k)} - 19S_{\sigma_6}^{(k)}), k = 1,2, \mu_1^* = \frac{-\mu_1}{2419200}, \mu_2^* = \frac{\mu_1}{345600},$$

$$\theta_k = \overline{\theta}_k = S_{\sigma_0}^{(k)} + \mu_k^* (715S_{\sigma_5}^{(k)} - 1755S_{\sigma_6}^{(k)} + 1485S_{\sigma_7}^{(k)} - 429S_{\sigma_8}^{(k)}) + \mu_k^{**} (23S_{\sigma_5}^{(k)} - 19S_{\sigma_6}^{(k)}),$$

$$k = 3,4, \mu_3^* = \frac{\mu_2}{483840}, \mu_3^{**} = \frac{-\mu_1}{115200}, \mu_4^* = \frac{-\mu_2}{96768}, \mu_4^{**} = \frac{\mu_1}{69120},$$

$$\theta_k = \overline{\theta}_k = S_{\sigma_0}^{(k)} + \mu_k^* (46189S_{\sigma_5}^{(k)} - 188955S_{\sigma_6}^{(k)} + 159885S_{\sigma_7}^{(k)} - 138567S_{\sigma_8}^{(k)} + 122265S_{\sigma_9}^{(k)} - 21879S_{\sigma_{10}}^{(k)})$$

$$+\overline{\mu}(715S_{\sigma_5}^{(k)} - 1755S_{\sigma_6}^{(k)} + 1485S_{\sigma_7}^{(k)} - 429S_{\sigma_8}^{(k)}) + \overline{\overline{\mu}}_k (23S_{\sigma_5}^{(k)} - 19S_{\sigma_6}^{(k)}), k = 5,6,$$

$$\mu_5^* = \frac{-\mu_5}{184320}, \mu_6^* = \frac{\mu_6}{61440}, \overline{\mu} = \frac{\mu_2}{48384}, \overline{\overline{\mu}}_5 = \frac{-\mu_1}{69120}, \overline{\overline{\mu}}_6 = \frac{\mu_1}{115200},$$

$$\theta_7 = \overline{\theta}_7 = S_{\sigma_0}^{(7)} + \frac{\mu_9}{24576}(1062347S_{\sigma_5}^{(7)} - 6084351S_{\sigma_6}^{(7)} + 15444891S_{\sigma_7}^{(7)} - 22309287S_{\sigma_8}^{(7)}$$

$$+19846665S_{\sigma_9}^{(7)} - 10567557S_{\sigma_{10}}^{(7)} + 3187041S_{\sigma_{11}}^{(7)} - 415701S_{\sigma_{12}}^{(7)}) - \frac{\mu_8}{61440}(46189S_{\sigma_5}^{(7)}$$

$$-188955S_{\sigma_6}^{(7)} + 159885S_{\sigma_7}^{(7)} - 138567S_{\sigma_8}^{(7)} + 122265S_{\sigma_9}^{(7)} - 21879S_{\sigma_{10}}^{(7)})$$

$$+\frac{\mu_2}{96768}(715S_{\sigma_5}^{(7)} - 1755S_{\sigma_6}^{(7)} + 1485S_{\sigma_7}^{(7)} - 429S_{\sigma_8}^{(7)}) - \frac{\mu_1}{345600}(23S_{\sigma_5}^{(7)} - 19S_{\sigma_6}^{(7)}).$$

In relation (3.5), we have $\sigma_i = i$ for $\alpha_{i,j}s$ and $\sigma_i = n+1-i$ for $\beta_{i,j}s$.

3.2 Extra boundary formulas

The space $Sp_8(\Delta)$ has $n+8$ dimensions, but the collocation equation (3.1) gives $n+p, 1 \le p \le 8$, nonlinear equations, thus we need $8-p$ extra equations to determine spline $S(x)$ uniquely. To obtain these extra boundary conditions we use the collocation equation at the boundary or its neighbor grid points. Let $E_x = \{x_0, x_1, x_2, x_3, x_{n-3}, x_{n-2}, x_{n-1}, x_n\}$, then we have the following system of $n+8$ equations for $1 \le p \le 8$.

$$\begin{cases} (L_p u = u^{(p)} - \phi)_{t_i}, & 1 \le i \le n, \\ (L_p u = u^{(p)} - \phi)_{x_r}, & x_r \in E_x, \\ (B_p u = \gamma_i)_{x_0, x_n}, & 0 \le i \le p-1. \end{cases} \tag{3.2.1}$$

3.3 Error estimation

Let $L_{p,i}^s$ and B_p^s be the perturbations of the operators L_p and B_p , respectively. In the case of $p = 8$, $L_{8,i}^s$ and B_8^s are those given above in equations (3.2)-(3.5). If $S(x)$ is the unique spline of degree eight to approximate the solution of problem (1.1)-(1.2) and satisfying (2.1)-(2.2), then the following relations hold:

$$\begin{cases} L_{p,i}^{s}S = O(h^{10}), & 1 \le i \le n, \\ L_{p,r}^{s}S = O(h^{10}), \\ B_{p}^{s}S = O(h^{10}). \end{cases} \tag{3.3.1}$$

4 Convergence Analysis

The proof of convergence analysis of the method is based on Green's function approach. The exact and spline solutions of the problem (1.1) which satisfy the boundary conditions $B_p u = \gamma$ are defined by $u^{(p)} = \psi_p$ and $\hat{S}^{(p)} = \nu_p$ respectively. Suppose that the boundary value problem $u^{(p)} = 0, p = 1(1)8$ subjected to boundary conditions $B_p u = 0$ has the unique solution. This means that there is a Green's function $G_p(x,t)$ for which $u(x)$ and $\hat{S}(x)$ can be obtained in the following forms:

$$u^{(i)}(x) = \int_a^b \frac{\partial^i G_p(x,t)}{\partial x^i}\psi_p(t)dt, \qquad \hat{S}^{(i)}(x) = \int_a^b \frac{\partial^i G_p(x,t)}{\partial x^i}\nu_p(t)dt, \qquad i = 0,\dots,p-1.$$

We need to introduce the following operators:

$$\Lambda_n : C[a,b] \to R^n, \Lambda_n g = [g(t_1),\dots,g(t_n)]^T, \mathbf{M}_n : R^n \to C[a,b], \pi$$
$$\mathbf{k}_p : C[a,b] \to C[a,b], \mathbf{k}_p g = \phi(x, \bar{G}_{p,0}(x), \bar{G}_{p,1}(x),\dots,\bar{G}_{p,j}(x)), \qquad j = p-1,$$

where $g \in C[a,b]$ and $\bar{G}_{p,i}(x) = \int_a^b \frac{\partial^i G_p(x,t)}{\partial x^i} g(t)dt, 0 \le i \le p-1$. Also we define an operator $\mathbf{R}_p$ with respect to the spline solution of the problem in the following form:

$$\mathbf{R}_p : C[a,b] \to C[a,b], \; \mathbf{R}_p g = \phi(x, Q_0\Lambda_n\bar{G}_{p,0}(x),\dots,Q_j\Lambda_n\bar{G}_{p,j}(x)), \qquad j = p-1,$$

where $g \in C[a,b], Q_0 = I_{n\times n}$ and $Q_p, 1 \le p \le 8$, are the coefficients matrices of $S_i^{(p)}$ in equation (3.3.1).

Lemma 4.1: If $p = \{p_{ij}\}$ be an $m \times m$ matrix and $p_{ii} \ge \Sigma_{j=1,i\ne j}^{m} |p_{ij}| + \grave{o}$, for $1 \le i \le m$, for $\grave{o} > 0$, then we have $\| p^{-1} \|_\infty \le \grave{o}^{-1}$.

Proof. See Lucas [24], Lemma 4 .

Lemma 4.2: If coefficient matrix of $S_i^{(p)}$ in the system of equations (3.3.1) is denoted by $Q_p, 1 \le p \le 8$ is nonsingular and $\| Q_p^{-1} \|_\infty$ is bounded.

Proof. For $p = 1,2,3$, matrix Q_p is strictly diagonally dominant and thus invertible. For $p = 4,5,6,7,8$ we can use elementary row operations, then Q_p can be converted to strictly diagonally dominant matrix, thus it is nonsingular, finally using Lemma 4.1 we can conclude that $\| Q_p^{-1} \|_\infty$, $1 \le p \le 8$ is finite.

By using the above notations in equations (1.1) and (3.3.1) we have:

$$\psi_p - \mathbf{k}_p \psi_p = (I - \mathbf{k}_p)\psi_p = 0, \tag{4.1}$$

and

$$Q_p \Lambda_n \hat{S}^{(p)} - \Lambda_n \mathbf{R}_p \nu_p = 0, \tag{4.2}$$

According to Lemma 4.2, Q_p is nonsingular, therefore we have

$$\mathbf{M}_n \Lambda_n \hat{S}^{(p)} - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p \nu_p = 0.$$

Since $\hat{S}^{(p)}(x)$ is a polynomial of order $8-p$, $\mathbf{M}_n \Lambda_n \hat{S}^{(p)}(x) \equiv \hat{S}^{(p)}(x)$ we have

$$\hat{S}^{(p)} - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p \nu_p = (I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)\nu_p = (I - p_n \mathbf{R}_p)\nu_p = 0, \tag{4.3}$$

where $p_n = \mathbf{M}_n Q_p^{-1} \Lambda_n$ is an operator from $C[a,b]$ into the continuous piecewise functions of order $8-p$ with break points t_i .

Lemma 4.3: Let $\{\Delta\}$ be a sequence of partitions of the interval $[a,b]$ with step size h . If $h \to 0$ as n increases then $p_n = \mathbf{M}_n Q_p^{-1} \Lambda_n$ converges to the identity operator uniformly.

Proof. Let $f \in C[a,b]$, we need to show that $| p_n f - f | \to 0$. We have

$$\| p_n f - f \| \le \| \mathbf{M}_n Q_p^{-1} \Lambda_n f - \mathbf{M}_n \Lambda_n f \| + \| \mathbf{M}_n \Lambda_n f - f \| .$$

Since the second term is of order $O(h^{9-p}), 1 \le p \le 8$ thus $\| \mathbf{M}_n \Lambda_n f - f \| \to 0$, so we have $\| p_n f - f \| \le \| \mathbf{M}_n Q_p^{-1} \Lambda_n f - \mathbf{M}_n \Lambda_n f \| \le \| \mathbf{M}_n \| \| Q_p^{-1} \| \ \| \Lambda_n f - Q_p \Lambda_n f \| \le C^* \| \Lambda_n f - Q_p \Lambda_n f \|$,

where C^* is a finite constant thus we have $\| p_n f - f \| \le C^* \| \Lambda_n f - Q_p \Lambda_n f \| \le C^{**} \omega(f, 13h)$,

Where $\omega(f, \epsilon) = sup\{| f(x + \delta) - f(x) | : x, x + \delta \in [a,b], | \delta | \le \epsilon\}$ is the modulus of continuity of $f(x)$. Since $f(x)$ is continuous function and $h \to 0$ we have $\omega(f, 13h) \to 0$. So this completes the proof.

Lemma 4.4: Under the hypotheses of Lemma 4.3, the operator sequence $p_n \mathbf{R}_p$ converges to $\mathbf{k}_p$ uniformly.

Proof. Let $f \in C[a,b]$ thus we have

$$\begin{aligned} &\| p_n \mathbf{R}_p f - \mathbf{k}_p f \| = \| \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p f - \mathbf{k}_p f \| \\ &\le \| \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p f - \mathbf{M}_n \Lambda_n \mathbf{k}_p f \| + \| \mathbf{M}_n \Lambda_n \mathbf{k}_p f - \mathbf{k}_p f \|, \\ &\le \| \mathbf{M}_n Q_p^{-1} \| \| \Lambda_n \mathbf{R}_p f - Q_p \Lambda_n \mathbf{k}_p f \| + O(h^{9-p}), \end{aligned} \tag{4.4}$$

Since $\| \mathbf{M}_n \|$ and $\| Q_p^{-1} \|$ are bounded thus for some finite constant C^{**} we have

$$\| (p_n \mathbf{R}_p - \mathbf{k}_p) f \| \leq C^{**} \| \Lambda_n \mathbf{R}_p f - Q_p \Lambda_n \mathbf{k}_p f \| \leq C^{**} \omega(f, \delta),$$

Where

$$\delta = max\{13h, \omega(\bar{G}_{p,0}(x), 21h), \omega(\bar{G}_{p,1}(x), 21h), ..., \omega(\bar{G}_{p,p-1}(x), 21h)\}. \tag{4.5}$$

Since $\bar{G}_{p,j}(x), 0 \leq i \leq p-1$ are continuous, $\omega(\bar{G}_{p,j}(x), 21h) \to 0, 0 \leq i \leq p-1$ as $h \to 0$. Now from (4.5) $\delta \to 0$. Finally since f is a continuous function and $\delta \to 0$ thus we have $\omega(f, \delta) \to 0$. To guarantee the uniqueness of the solution of (1.1)-(1.2) at least in a small neighborhood of an isolated solution, we need to state the following Theorem that is treated in [13].

Theorem 4.1: Suppose that $u(x)$ is a solution BVPs (1.1)-(1.2) and the continuous functions $\phi(x, z_0, z_1, ..., z_{p-1})$ and $\frac{\partial^i}{\partial z_i} \phi(x, z_0, z_1, ..., z_{p-1}), (0 \leq i \leq p-1)$, are defined and continuous in the following region: $a \leq x \leq b, | z_i - u^{(i)}(x) | \leq \delta^*, \quad (0 \leq i \leq p-1, \delta^* > 0)$. also suppose that the homogeneous equation $u^{(p)} = 0$, subjected to boundary conditions (1.2) has only the trivial solution. Consider a sequence of partition $\{\Delta\}$ of $[a,b]$ such that $h \to 0$. If the linear homogenous equation,

$$u^{(p)}(x) - \sum_{i=0}^{p-1} \frac{\partial^i}{\partial z_i} \phi(x, z_0, z_1, ..., z_{p-1}) u^{(i)}(x) = 0,$$

subjected to the boundary conditions (1.2) has only a trivial solution, then there exists a $\sigma > 0$, such that $u(x)$ is the unique solution of (1.1)-(1.2) in the sphere $\| w - u^{(p)} \| \leq \sigma$, further for sufficiently large n there exists a unique spline $\hat{S} \in Sp_8(\Delta)$ satisfying (1.1)-(1.2) such that $\| \hat{S}^{(p)} - u^{(p)} \| \leq \sigma$, and $\hat{S}(x)$ and its derivatives through $p-1$, converges to $u(x)$ and its derivatives of corresponding orders.

Proof. The proof is similar to that in Russell and Shampine, ([13], Theorem 5).

Now we want to verify and prove the main convergence theorem which gives the error bounds and the orders of convergence of the purposed method theoretically.

Theorem 4.2: Consider a sequence of partition $\{\Delta\}$ of interval $[a,b]$ such that the step size $h \to 0$. Suppose that $\hat{S}(x)$ is a spline of degree eight approximation to problems (1.1)-(1.2) then under the assumptions of Theorem 4.1, the following error bounds hold for the presented method,

$$\| (u - \hat{S})^{(j)} \| = O(h^{9-j}), j = 0, ..., 7, \; | (u - \hat{S})_i^{(j)} | = O(h^{10}), j = 0,$$
$$| (u - \hat{S})_i^{(j)} | = O(h^8), j = 1, 2, \; | (u - \hat{S})_i^{(j)} | = O(h^6), j = 3, 4,$$
$$| (u - \hat{S})_i^{(j)} | = O(h^4), j = 5, 6, \; | (u - \hat{S})_i^{(j)} | = O(h^2), j = 7.$$

Proof. We assume that $S(x) \in Sp_8(\Delta)$, be the unique spline interpolant of u as in Theorem 2.1 and ψ_p and ν_p are the exact and spline solutions of the problem respectively. We consider the following problem $S^{(p)} = \chi,\ B_p^s S = O(h^{10})$. According to the hypotheses there is a unique solution to the problem $u^{(p)} = 0, B_p u = 0$ thus there exists a polynomial $\xi(x)$ of order $p-1$ such that

$$B_p^s \xi = B_p^s S = O(h^{10}), \quad \| \xi^{(k)} \| = O(h^{10}),\ k = 0, ..., p-1. \tag{4.6}$$

From solvability of $(S-\xi)^{(p)} = \chi, B_p^s(S-\xi) = 0$ we have

$$(I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)(S^{(p)} - \xi^{(p)}) = \mathbf{M}_n Q_p^{-1}(Q_p \Lambda_n - \Lambda_n \mathbf{R}_p)(S-\xi)^{(p)},$$

then using (3.3.1) and the boundness of $\| \mathbf{M}_n \|$ and $\| Q_p^{-1} \|$ we get

$$(I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)(S^{(p)} - \xi^{(p)}) = \mathbf{M}_n Q_p^{-1}(O(h^{10})) = O(h^{10}). \tag{4.7}$$

Subtracting equations (4.3) and (4.7) we have

$$(I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = O(h^{10}),$$

and equivalently

$$(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = p_n \mathbf{R}_p (S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = O(h^{10}). \tag{4.8}$$

It is known that the operator $\mathbf{R}_p$ is continuously differentiable in an area about u (see [13]) thus we can write (4.8) in the following integral equation form

$$(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) =$$
$$p_n \left(\int_0^1 (\mathbf{R}_p^{'}[\hat{S}^{(p)} + t(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)})]dt \right)(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = O(h^{10}), \tag{4.9}$$

where $\{\vartheta_n\} = p_n \left(\int_0^1 (\mathbf{R}_p^{'}[\hat{S}^{(p)} + t(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)})]dt \right)$ is a sequence of linear operators converging to $\mathbf{R}_p^{'}(u^{(p)})$. Thus we have

$$(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = \vartheta_n (S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) + O(h^{10}).$$

Finally since $(I - \vartheta_n)^{-1}$ exists and its norm is uniformly bounded we have

$$\| (S - \xi - \hat{S})^{(p)} \|_\infty = O(h^{10}). \tag{4.10}$$

Now according to the hypotheses the problem $(S - \xi - \hat{S})^{(p)} = r, B_p^s (S - \xi - \hat{S}) = 0$ is uniquely solvable thus we have via Green's function notation

$$(S - \xi - \hat{S})^{(i)} = \int \frac{\partial^i G_p(x,t)}{\partial x^i}(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)})(t)dt, \ i = 0,...,p-1, \tag{4.11}$$

which implies

$$\| (S - \xi - \hat{S})^{(i)} \|_\infty = O(h^{10}), \ i = 0,...,p-1. \tag{4.12}$$

Now using the triangular inequality we have

$$\| (u - S)^{(i)} \| \le \| (u - \hat{S})^{(i)} \| + \| (S - \hat{S})^{(i)} \| + \| \xi^{(i)} \|, \ i = 0,...,p-1,$$

then using equations (3.3.1) and (4.6) and Theorem 2.1, and the proof is completed.□

5 Illustrative Examples

In this section we consider five boundary value problems of various orders with appropriate boundary conditions, to demonstrate the efficiency and applicability of the presented method. We compared our numerical results with the results in [5-8, 22, 25-28] and the results are tabulated in Tables 1-7. The computational costs are tabulated. All programs run in software mathematica 10. The initial guess that we use for solving the arising algebraic systems is zero.

Example 1. Consider the following linear eighth order boundary value problem

$$u^{(8)}(x) = -xu(x) - e^x(48 + 15x + 2x^3), \quad 0 \le x \le 1$$
$$u(0) = u(1) = 0, u'(0) = 1, u'(1) = -e, u''(0) = 0, u''(1) = -4e, u^{(3)}(0) = -3, u^{(3)}(1) = -9e.$$

The exact solution for this problem is $u(x) = x(1-x)e^x$. First we solve this problem with step size $h = \frac{1}{30}$ and compared the errors in those given points in [5-7,25]. These results are tabulated in Table 1 and Fig. 1, which show that the maximum absolute errors in the solutions of our method are considerable less than those methods in [5-7,25]. Moreover we solve this problem for various values of $h = \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}$ and the maximum absolute errors in the solutions. CPU times are listed in Table 2. In this table E_i means, $E_i = \| u^{(i)} - \hat{S}^{(i)} \|_\infty, 1 \le i \le 7$, and O_i means the order of convergence of ith derivatives of u. This table also verified the accuracy nature of our method.

Table 1. Numerical results for example 1 with $h = \frac{1}{30}$

x	**Convergence method**	**Method in [25]**	**Method in [7]**	**Method in [6]**	**Method in [5]**
0.1	4.16(-16)	3.89(-15)	1.63(-10)	5.62(-10)	3.73(-9)
0.2	2.81(-14)	1.45(-14)	1.63(-9)	4.88(-9)	6.61(-9)
0.3	1.92(-14)	1.04(-14)	4.90(-9)	1.37(-8)	2.33(-8)
0.4	3.31(-13)	4.37(-13)	8.46(-9)	2.29(-8)	5.17(-8)
0.5	8.84(-13)	6.20(-13)	1.01(-8)	2.71(-8)	9.76(-8)
0.6	3.63(-13)	4.02(-13)	8.68(-9)	2.38(-8)	1.78(-6)
0.7	1.47(-14)	2.07(-12)	5.15(-9)	1.49(-8)	4.12(-6)
0.8	2.43(-14)	2.66(-12)	1.76(-9)	5.54(-9)	1.83(-4)

Table 2. Numerical results for example 1 with various values of h

h	**1/18**	**1/36**	**1/72**	**1/144**
E_0, O_0	2.2(-16),-	1.9(-19),10.2	1.8(-22),10.1	1.7(-25),10.1
E_1, O_1	8.7(-16),-	8.5(-19),9.9	8.4(-22),9.9	8.3(-25),9.9
E_2, O_2	6.7(-15),-	4(-17),6.7	6.2(-19),6.6	6.0(-21),6.6
E_3, O_3	7.8(-12),-	7.5(-14),6.7	7.3(-16),6.6	7.1(-21),6.6
E_4, O_4	4.3(-11),-	4.1(-13),6.7	3.9(-15),6.7	3.7(-17),6.7
E_5, O_5	1.3(-7),-	1.1(-9),6.8	1.0(-11),6.7	1.1(-13),6.5
E_6, O_6	4.3(-7),-	4.1(-9),6.7	3.8(-11),6.7	3.6(-13),6.7
E_7, O_7	3.1(-7),-	3.0(-8),3.4	2.8(-9),3.4	3.1(-10),3.2
CPU times	0.157	0.221	0.314	0.561

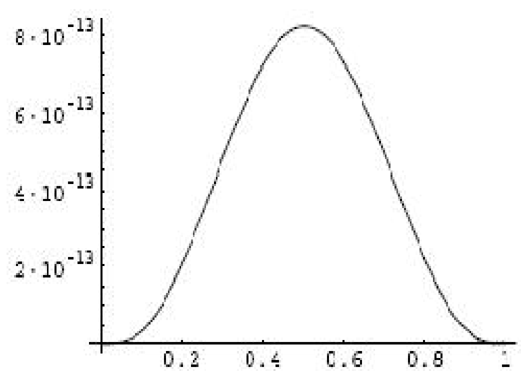

Fig. 1. Absolute error between exact and approximate solution (n = 30)

Example 2. Consider the following nonlinear eighth order boundary value problem

$$u^{(8)}(x) = u^2(x)e^{-x}, \quad 0 \le x \le 1$$

along with the boundary conditions,

$$u(0) = u'(0) = u''(0) = u^{(3)}(0) = u^{(4)}(0) = 1, \quad u(1) = u'(1) = u''(1) = e.$$

With the exact solution $u(x) = e^x$. We solve this problem with step size $h = \frac{1}{30}$. The results are compared with those mentioned in [5,26]. The maximum absolute errors in the solutions are tabulated in Table 3. Our results are compared with the results obtained by Adomian decomposition method and Variational iteration decomposition method, which show the applicability of our method computationally.

Table 3. Numerical results for example 2 with $h = \frac{1}{30}$

x	Convergence method	Method in [5]	Method in [26]
0.1	5.21(-13)	2.34(-4)	1.27(-5)
0.2	2.62(-14)	2.54(-5)	2.43(-5)
0.3	7.04(-14)	3.02(-5)	3.35(-5)
0.4	1.87(-11)	5.26(-5)	3.94(-5)
0.5	3.54(-11)	8.39(-5)	4.16(-5)

Example 3. Consider the following nonlinear eighth order boundary value problem

$$u^{(8)}(x) + 3u^{(7)} + u^{(6)}(x) + u'^2(x)e^{4u(x)} - 4u''(x)u^2(x) + u'''^2(x)e^{2x} = -36e^{-2x}, 0 \le x \le 1$$
$$u(0) = 1, u(1) = e^{-2}, u'(0) = -2, u'(1) = -2e^{-2},$$
$$u''(0) = 4, u''(1) = 4e^{-2}, u^{(3)}(0) = -8, u^{(3)}(1) = -8e^{-2}.$$

The exact solution of this problem is $u(x) = e^{-2x}$. We solve this problem with step size $h = \frac{1}{30}$ for various of points and compared the errors in those special points given in [28]. Results are tabulated in Table 4. Our results are compared with the results obtained by Galerkin method. The data in table verified that our method is more accurate. Finally we solve this problem with $h = \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}$, the maximum absolute errors in the solutions and CPU times are tabulated in Table 5. This table shows that the orders of convergence in applications agree with those we obtained theoretically.

Table 4. Numerical results for example 3 with $h = \frac{1}{30}$

x	Convergence method	Method in [28]
0.1	4.32(-14)	7.98(-6)
0.2	7.83(-12)	2.18(-5)
0.3	4.22(-11)	2.09(-5)
0.4	6.64(-9)	2.87(-5)
0.5	5. 42(-9)	2.68(-5)
0.6	9.27(-8)	1.69(-5)
0.7	6.53(-8)	1.15(-5)
0.8	4. 37(-7)	4.79(-6)
0.9	5.23(-6)	1.82(-6)

Table 5. Numerical results for example 3 with various values of h

h	**1/18**	**1/36**	**1/72**	**1/144**
E_0, O_0	5.1(-16),-	4.9(-19),10	4.8(-22),9.9	4.7(-25),9.9
E_1, O_1	6.9(-15),-	6.7(-18),10	6.5(-21),10	6.3(-24),10
E_2, O_2	2.4(-14),-	2.3(-17),10	2.4(-20),9.9	2.7(-23),9.7
E_3, O_3	7.6(-13),-	7.3(-15),6.7	7.1(-17),6.6	6.9(-19),6.6
E_4, O_4	3.2(-11),-	3.1(-13),6.6	2.8(-15),6.7	2.7(-17),6.7
E_5, O_5	9.7(-9),-	8.8(-11),6.8	8.6(-13),6.7	8.1(-15),6.7
E_6, O_6	3.2(-9),-	3.4(-10),3.2	3.0(-11),3.5	2.6(-12),3.5
E_7, O_7	1.7(-8),-	1.6(-9),3.4	1.5(-10),3.4	1.1(-11),3.4
CPU times	0.266	0.325	0.500	0.926

Example 4. Consider the following nonlinear seventh order boundary value problem

$$u^{(7)}(x)+u^{(4)}(x)-u(x)e^{u(x)}=e^{x}((-4(-3+x)+e^{(-e^{x}(x-1)\cos x)}(x-1))\cos x-8(5+x)\sin x),0\le x\le 1$$

subject to the boundary conditions,

$$u(0)=1,u'(0)=u(1)=0,u'(1)=-e\cos 1,u''(0)=u^{(3)}(0)=-2,u''(1)=-2e\cos 1+2e\sin 1.$$

With the exact solution $u(x)=e^{x}(1-x)\cos x$. We solve this problem with step size $h=\frac{1}{30}$ and compared the errors in those special points given [8]. The maximum absolute errors in the solutions are tabulated in Table 6 and Fig. 2. Our results are compared with the results obtained Reproducing kernel method. The results in this table show the applicability of our method computationally.

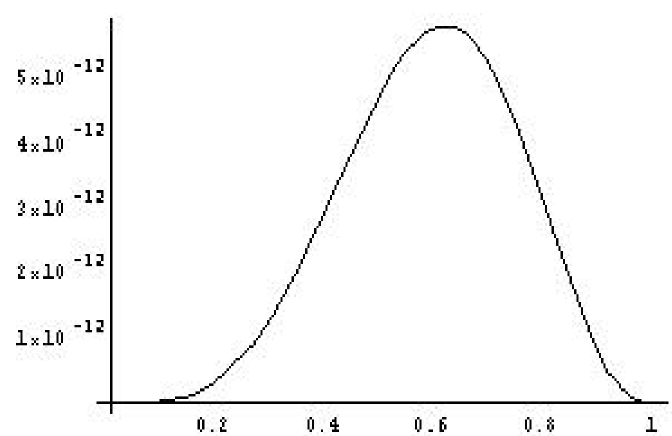

Fig. 2. Absolute error between exact and approximate solution (n = 30)

Table 6. Numerical results for example 4 with $h = \frac{1}{30}$

x	Convergence method	Method in [8]
0.125	3.51(-14)	4.74(-10)
0.250	2.09(-12)	5.20(-9)
0.375	4.29(-12)	1.53(-8)
0.500	5.01(-12)	2.45(-8)
0.625	4.04(-12)	2.53(-8)
0.750	2.12(-12)	1.56(-8)

Example 5. Consider the following nonlinear sixth order boundary value problem

$$u^{(6)}(x) = u^2(x)e^{-x}, \quad 0 \le x \le 1$$

$$u(0) = u''(0) = u^{(4)}(0) = 1, \quad u(1) = u''(1) = u''(1) = u^{(4)}(1) = e.$$

With the exact solution $u(x) = e^x$. We solve this problem with step size $h = \frac{1}{30}$. The results are compared with those mentioned in [22, 27]. The maximum absolute errors in the solutions are tabulated in Table 7. Our results are compared with the results obtained by sextic spline collocation method and Variational iteration method. The results in this table verified that our method is more accurate.

Table 7. Numerical results for Example 5 with $h = \frac{1}{30}$

x	Convergence method	Method in [22]	Method in [27]
0.1	2.07(-17)	2.08(-15)	1.23(-4)
0.2	8.33(-16)	8.39(-15)	2.35(-4)
0.3	6.03(-16)	5.94(-15)	3.25(-4)
0.4	5.28(-16)	1.56(-14)	3.85(-4)
0.5	4.06(-16)	2.53(-14)	4.08(-4)
0.6	3.14(-16)	3.18(-14)	3.91(-4)
0.7	2.24(-16)	3.24(-14)	3.36(-4)
0.8	1.35(-16)	2.60(-14)	2.45(-4)
0.9	1.07(-16)	1.38(-14)	1.29(-4)

6 Conclusion

In this paper, a new tenth order super convergence method based on eight degree B-spline has been developed for solution of higher order boundary value problems. The results shown that the accuracy of the computed solutions are in good agreement with the analytical solutions. The method is easy to apply, and can be applied to similar problems that arise in engineering and sciences, easily. The good accuracy of the proposed method has been tested and shown on some linear and nonlinear problems. Computed solutions are compared with references [5-8,22,25-28]. It is observed that the absolute error is the solution are considerable accurate. Mathematica software is used for all computational work.

Competing Interests

Authors have declared that no competing interests exist.

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