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Research Article

Existence of Traveling Wave Fronts for a Generalized Nonlinear Schrodinger Equation

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In the presented paper, a generalized nonlinear Schrodinger equation without delay convolution kernel and with special delay convolution kernel is investigated. By using the geometric singular perturbation theory, the existence of traveling wave fronts is proved. Firstly, we show that such traveling wave fronts exist without delay by non-Hamiltonian qualitative analysis. Then, for the generalized nonlinear Schrodinger equation with a special local strong delay convolution kernel, the desired heteroclinic orbit is obtained by using the Fredholm theory.

1. Introduction

In recent decades, there are two important integrable models of nonlinear mathematical physics, which are the well-known nonlinear Schrodinger (NLS) equation [1]

$$iu_t + au_{xx} + bu|u|^2 = 0 \tag{1}$$

and the derivative nonlinear Schrodinger (DNLS) equation

$$iu_t + au_{xx} + bi\left(u|u|^2\right)_x = 0, \tag{2}$$

where $i = \sqrt{-1}$, u is a complex valued function of the spatial coordinate x and the time t, and $a, b \in R$ are dispersion coefficient and Landau coefficient, respectively. $|u|^2 = uu^*$, u^* is the complex conjugate of u. Equations (1) and (2) model the propagation of intense laser beams in isotropic media and admits solutions that become infinitely large after a finite propagation distance. In the case of ultrashort laser pulses, temporal effects (such as dispersion) can become important. In addition, at sufficiently high intensities, the electric field ionizes the medium, resulting in plasma formation [2, 3]. This, in turn, leads to changes in the optical properties of the medium, which are unaccounted for in (1). The nonlinear Schrodinger equation model also neglects high-order nonlinear polarizations by the medium. Traveling wave solutions of this equation and a variety of generalizations have been widely studied for a long time [4–6]. In reference [7], they theoretically address the existence of traveling wave solutions of the following delay nonlinear Schrodinger equation:

$$iu_t + u_{xx} + (f * u)|u|^2 - \gamma u|u|_x^2 = 0. \tag{3}$$

$u|u|_x^2$ means nonlinear response delay term [8] and parameter $\gamma > 0$. They devoted to study of traveling waves of nonlinear Schrodinger equation with distributed delay by applying geometric singular perturbation theory, differential manifold theory, and the regular perturbation analysis. The existence of a homoclinic connection and the periodic orbits were established. In reference [9], some exact solutions of the two-component general nonlinear Schrodinger equation are obtained by using the general Darboux transformation, including rogue-wave solution, breather solution, and breather-rogue-wave interaction.

In this paper, we shall use the geometric singular perturbation theory [10–13] to investigate the following generalized nonlinear Schrodinger (GNLS) equation

$$iu_t + u_{xx} + (f * u)|u|^n - \gamma u|u|_x^2 = 0. \quad (4)$$

where $n \geq 1$ is an integer and $f * u$ represents a convolution in the spatial variable. When $n > 2$, it means there is a time delay for the higher order nonlinear Landau term. When $n = 2$, equation (4) reduces to equation (3). We should remark that time delay does implicitly play a significant role in the dynamical behaviors of (4), as the nonlinear Landau term involves the size of this time lag. For equation (4), we analyze the corresponding ordinary differential equation without delay, with a nonlocal weak generic delay convolution kernel, respectively, and then prove the existence of traveling wave fronts by using geometric singular perturbation theory.

The remaining part of this paper is organized as follows. The geometric singular perturbation theory is presented in Section 2. In Section 3, equation (4) is investigated in two cases: without delay, with a local strong generic delay kernel. The existence of traveling wave fronts for equation (4) is obtained by using geometric singular perturbation theory, Fredholm theory. It is a simplified conclusion in Section 4.

2. Preliminaries

Firstly, we introduce the following result on invariant manifolds which is due to Fenichel [14, 15].

Lemma 1. *For the standard fast-slow system,*

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \quad (5)$$

where $0 < \varepsilon \ll 1$ is a real parameter, $x = (x_1, x_2, \cdots, x_k)^T \in R^k$, $y = (y_1, y_2, \cdots, y_l)^T \in R^l$, $k + l = n.x, y$ are C^∞ on the set $U \times V$, where $U \subset R^n$ and V is an open interval containing 0. Assume that for $\varepsilon = 0$, the system has a compact, normal hyperbolic manifold of critical point M_0, which is contained in the set $f(x, y, 0) = 0$. The manifold M_0 is hyperbolic normally if the linearization of (5) at each point in M_0 has exactly l eigenvalues with zero real part, where l is the dimension of the center point. Therefore, for any $0 < r < +\infty$, if $\varepsilon > 0$ and sufficiently small, there exists a manifold M_ε, which is locally invariant under the flow of (5), which is C^r in x, y and ε. What is more, $M_\varepsilon = \{(x, y): x = h^\varepsilon(y)\}$ for some C^r function $h^\varepsilon(y)$ and y in some compact K. There exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that lie within $O(\varepsilon)$, and are diffeomorphic to $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$, respectively.

Definition 2. A set M is locally invariant under the flow from (5) if it has neighborhood V so that no trajectory can leave M without also leaving V. In other words, it is locally invariant if for all $x \in M$, $x \cdot [0, t] \subset V$ implies that $x \cdot [0, t] \subset M$, similarly with $[0, t]$ replaced by $[t, 0]$, when $t < 0$, where the notation $x \cdot t$ is used to denote the application of a flow after time t to the initial condition x.

With a change of time-scale $\tau = \varepsilon t, \doteq d/d\tau$, system (5) can be reformulated to

$$\begin{cases} \varepsilon \dot{x}(t) = f(x, y, \varepsilon), \\ \dot{y}(t) = g(x, y, \varepsilon). \end{cases} \quad (6)$$

When $\varepsilon \neq 0$, systems (5) and (6) are equivalent, system (5) is called the fast system, and (6) is called the slow system. Each of the scalings is naturally associated with a limit as $\varepsilon \longrightarrow 0$. These limits of (5) and (6) are, respectively, given by

$$\begin{cases} x'(t) = f(x, y, 0), \\ y'(t) = 0, \end{cases} \quad \begin{cases} 0 = f(x, y, 0), \\ \dot{y}(t) = g(x, y, 0). \end{cases} \quad (7)$$

The former is called the layer problem and the latter is called the reduced system.

3. Existence of Traveling Wave Fronts

Traveling wave front solution $u(x, t) = \phi(x - ct)$ is strictly monotonic with respect to $\xi = x - ct$ and globally asymptotically stable with phase shift. For equation (4), on certain parametric conditions, there exist traveling wave fronts, which satisfy $\phi(-\infty) \neq \phi(+\infty)$. In fact, it means there are two steady states of the equation, In this section, the system reduction is presented. Then, we will establish the existence of traveling wave fronts for equation (4) in two cases: without delay and with a local strong generic delay kernel, respectively.

3.1. The Model without Delay. For delay generalized nonlinear Schrodinger equation (4), the traveling wave form with $u(x, t) = \phi(\xi)e^{i(kx-\alpha t)}$, $\xi = x - ct$, and $c > 0$, where ϕ is real-valued function and represents the amplitude of the traveling wave with wave number $k > 0$ and frequency $\alpha > 0$. By the property of the kernel function f, when without delay, equation (4) reduces to

$$iu_t + u_{xx} + u|u|^n - \gamma u|u|_x^2 = 0. \quad (8)$$

For a given constant $c > 0$, substituting $u(x, t) = \phi(\xi)$ into (8), the real part and the imaginary part of the nondelay equation are given by

$$\begin{gathered} (-\alpha + k^2)\phi + \phi'' + \phi^{n+1} - 2\gamma\phi^2\phi' = 0, \\ -c + 2k = 0. \end{gathered} \quad (9)$$

Then, $k = c/2$. Equation (9) is equivalent to the following system of first-order equations:

$$\begin{cases} \phi' = y, \\ y' = \left(k^2 - \alpha\right)\phi - \phi^{n+1} + 2\gamma\phi^2 y, \end{cases} \tag{10}$$

where $' = d/d\xi$. Obviously, system (10) is a non-Hamiltonian system. Assume that $k^2 - \alpha > 0$; it is not difficult to know that (10) has two equilibria $E_0(0,0)$ and $E_1(\sqrt[n]{k^2-\alpha},0)$, and $E_0(0,0)$ is a center; $E_1(\sqrt[n]{k^2-\alpha},0)$ is a node when $k^2 - \alpha > 1$, $\gamma > 2$, and $1 \le n < 4$.

In order to verify the existence of heteroclinic orbit between $E_0(0,0)$ and $E_1(\sqrt[n]{k^2-\alpha},0)$, we confine $y > 0$. For a suitable value $\lambda > 0$, the triangular

$$\Omega = \left\{(\phi, y)\colon 0 \le \phi \le \sqrt[n]{k^2-\alpha}, 0 \le y \le \lambda\phi\right\} \tag{11}$$

is positive invariant. Let $\vec{F}$ be the vector defined by the right-hand side system (10) and $\vec{n}$ be the inward pointing normal on the boundary of Ω. On the slide $\phi = \lambda$y, $0 \le \phi \le \sqrt[n]{\mathrm{k}^2 - \alpha}$, setting that $\vec{n} = (\lambda,-1)$, it obtains

$$\begin{aligned} \vec{F}\cdot\vec{n} &= \left(y, \left(k^2-\alpha\right)\phi - \phi^{n+1} + 2\tau\phi^2 y\right)\cdot(\lambda,-1)|(\phi,\lambda\phi) \\ &= \lambda^2\phi - \left[\left(k^2-\alpha\right)\phi - \phi^{n+1} + 2\gamma\lambda\phi^3\right] \\ &\le \left(\lambda^2 + \alpha - k^2\right)\phi - 2\gamma\lambda\phi^3 \le \left(\lambda^2 + \alpha - k^2\right)\phi. \end{aligned} \tag{12}$$

When $0 < \lambda \le \sqrt{k^2-\alpha}$, it is obvious that $\lambda^2 + \alpha - k^2 \le 0$; it implies that $\vec{F}\cdot\vec{n} \le 0$. Thus, one branch of the stable manifold at $E_1(\sqrt[n]{k^2-\alpha},0)$ always stays in the region Ω and joins $E_1(\sqrt[n]{k^2-\alpha},0)$ into the origin. It implies that the desired heteroclinic orbit exists. Consequently, we obtain the following theorem.

Theorem 3. *Assume that $\tau > 0$ is small sufficiently, $k^2 - \alpha > 1$, $\gamma > 2$, and $1 \le n < 4$, then on the (ϕ, y) phase plane for system (10), there exists a heteroclinic orbit between $E_0(0,0)$ and $E_1(\sqrt[n]{k^2-\alpha},0)$, which is confined on $y > 0$. The traveling wave fronts $u(x,t) = \phi(x+ct)$ for system (9) is strictly decreasing and satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = \sqrt[n]{k^2-\alpha}$.*

3.2. The Model with Nonlocal Delay. From Subsection 3.1, the existence of heteroclinic orbits is shown, so we shall verify the one connecting $E_0(0,0)$ and $E_1(\sqrt[n]{k^2-\alpha},0)$ exists. The delay to be incorporated in a way that allows for associated spatial averaging due to the diffusion. This idea was first introduced by Britton [16]. The existence of traveling wave fronts for equation (4) with a local strong generic kernel is considered by using geometric singular perturbation theory, the Fredholm theory. The convolution $f * u$ is denoted by

$$(f * u)(x,t) = \int_{-\infty}^{t} f(t-s)u(x,s)ds. \tag{13}$$

Because u is a complex valued function, the kernel f can be defined as a complex valued function, i.e., the kernel $f : [0,+\infty) \longrightarrow \mathscr{C}$, which satisfies the following normalization condition:

$$\begin{gathered} \int_0^\infty |f(t)|dt = 1, \\ t|f(t)| \in L^1((0,+),\mathscr{R}), \end{gathered} \tag{14}$$

such that the kernel does not affect the spatially uniform steady-state. The average delay for the distributed delay kernel $f(t)$ is defined as $\tau = \int_0^\infty t|f(t)|dt$. In particular, there are the following nonlocal weak and strong generic delay kernels

$$\begin{aligned} f(t) &= \frac{1}{\tau}e^{-(t/\tau)}, \\ f(t) &= \frac{t}{\tau^2}e^{-(t/\tau)}, \end{aligned} \tag{15}$$

where the parameter $\tau > 0$ measures the average time delay. Here, we discuss equation (4) with a local strong generic kernel. Equation (4) changes to

$$iu_t + u_{xx} + |u|^n \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-((t-s)/\tau)} u(x,t)ds - \tau u|u|_x^2 = 0. \tag{16}$$

Suppose that the traveling wave front wave solution of (16) is the form $u(x,t) = \phi(\xi)e^{i(kx-\alpha t)}$, with $\xi = x - ct$, $c > 0$ is the wave speed. The real part and imaginary part of corresponding ordinary differential equation is

$$\left(-\alpha + k^2\right)\phi + \phi'' + \phi^n\omega - 2\sqrt[n]{k^2-\alpha}\phi^2\phi' = 0 - c + 2k = 0, \tag{17}$$

where $' = d/d\xi$ and

$$\omega(x,t) = (f * \phi)(\xi) = \int_0^\infty \frac{s}{\tau^2} e^{-(s/\tau)}\phi(\xi)ds. \tag{18}$$

By direct computation, it obtains

$$\frac{d\omega}{d\xi} = \frac{1}{c\tau}(\varphi - \omega), \tag{19}$$

where

$$\varphi = \int_0^\infty \frac{1}{\tau} e^{-(s/\tau)}\phi(\xi)ds. \tag{20}$$

Then, it has

$$\frac{d\varphi}{d\xi} = \frac{1}{c\tau}(\phi - \varphi). \tag{21}$$

Thus, equation (17) is equivalent to a four-dimensional system as follows:

$$\begin{cases} \phi' = y, \\ y' = (k^2-\alpha)\phi - \phi^n\omega - 2\sqrt[n]{k^2-\alpha}\phi^2 y, \\ c\tau\omega' = \varphi - \omega, \\ c\tau\varphi' = \phi - \varphi, \end{cases} \tag{22}$$

which is under the boundary conditions $\phi(-\infty)=0$, $\phi(+\infty)=\sqrt[n]{k^2-\alpha}$,$\varphi(\pm\infty)=0$. From the third and the fourth equation in (22), when $\tau \longrightarrow 0$, it has $\omega \longrightarrow \varphi$ and $\varphi \longrightarrow \phi$. At this limit, (22) can reduce to the nondelay model (9), which holds two equilibria $E_0(0,0)$, $E_1(\sqrt[n]{k^2-\alpha},0)$. For $\tau>0$, in the (ϕ,y,ω,φ) phase space, (22) has two equilibria: $(\phi,y,\omega,\varphi)=(0,0,0,0)$ and $(\phi,y,\omega,\varphi)=(\sqrt[n]{k^2-\alpha},0,\sqrt[n]{k^2-\alpha},\sqrt[n]{k^2-\alpha})$. The small parameter $\tau>0$ plays the delay in the original system, and (22) reduces to a regular perturbed system. Therefore, to show the existence of traveling wave fronts for equation (4), we need to show the existence of traveling wave fronts for system (22). Note that when $\tau=0$, (22)) does not define a dynamical system in R^4; therefore, we use the transformation $\xi=\varepsilon\zeta$; the system (22) can be rewritten as

$$\begin{cases} \dot\phi = \tau y, \\ \dot y = \tau\left[(k^2-\alpha)\phi - \phi^n\omega - 2\sqrt[n]{k^2-\alpha}\phi^2 y\right], \\ c\dot\omega = \varphi - \omega, \\ c\dot\varphi = \phi - \varphi, \end{cases} \tag{23}$$

where $\cdot$ denotes the derivative with respect to ζ. System (23) is the fast system, they are equivalent when $\tau>0$. When $\tau=0$, then the flow of the slow system is defined to a set

$$M_0 = \left\{(\phi,y,\omega,\phi)\in R^4 : \omega=\varphi, \varphi=\phi\right\}, \tag{24}$$

which is a two-dimensional invariant manifold of (22) with $\tau=0$. In order to find a two-dimensional invariant manifold for sufficiently small $\tau>0$ by using geometric singular perturbation theory, we have to verify that the variant manifold is normally hyperbolic. Therefore, we find an invariant manifold M_τ of system (23) when $\tau>0$, which is closed to M_0. The restriction of (23) to this invariant manifold M_τ yields a two-dimensional system, since the linearized matrix of (23) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} & \frac{1}{c} \\ \frac{1}{c} & 0 & 0 & -\frac{1}{c} \end{pmatrix}. \tag{25}$$

It is easy to obtain the eigenvalues are $0, 0, -1/c, -1/c$; then, the number of the eigenvalues with zero real part is equal to $\dim M_0$ and the other eigenvalues are hyperbolic. Thus, the slow manifold M_0 is normally hyperbolic. From geometric singular perturbation theory, it is obvious that there exists a submanifold M_τ of the perturbed system (23) of R^4 for sufficiently small $\tau>0$, which can be written as

$$M_\tau = \left\{(\phi,y,\omega,\varphi)\in R^4 : \omega=\varphi+g(\phi,y\tau), \varphi=\phi+h(\phi,y,\tau)\right\}, \tag{26}$$

where h, g are smooth functions defined on a compact domain and satisfy

$$g(\phi,y,0)=0, h(\phi,y,0)=0. \tag{27}$$

Thus, the functions g and h can be expanded into the form of a Taylor series about τ as follows:

$$\begin{aligned} g(\phi,y,\tau) &= \tau g_1(\phi,y,\tau) + \tau^2 g_2(\phi,y,\tau) + O(\tau^3), \\ h(\phi,y,\tau) &= \tau h_1(\phi,y,\tau) + \tau^2 h_2(\phi,y,\tau) + O(\tau^3). \end{aligned} \tag{28}$$

Substituting $\omega=\varphi+\mathrm{g}(\phi,y,\tau)$ and $\varphi=\phi+h(\phi,y,\tau)$ into the slow system (22), we have

$$\begin{aligned} &\frac{1}{c\tau}(\phi-\varphi) + \tau\left\{\frac{\partial g_1}{\partial\phi}y + \frac{\partial g_1}{\partial y}\left[(k^2-\alpha)\phi - \phi^n\omega - 2\sqrt[n]{k^2-\alpha}\phi^2 y\right]\right\} \\ &\quad - 2\gamma\phi^2 y = \frac{1}{c}(-g_1-\tau g_2), y \\ &\quad + \tau\left\{+\frac{\partial h_1}{\partial\phi}y + \frac{\partial h_1}{\partial y}\left[(k^2-\alpha)\phi - \phi^n\omega - 2\sqrt[n]{k^2-\alpha}\phi^2 y\right]\right\} \\ &\quad = \frac{1}{c}(-h_1-\tau h_2), \end{aligned} \tag{29}$$

By comparing coefficients of τ with each degree, we obtain

$$\begin{aligned} g_1 &= cy, g_2 = -2c^2\left[(k^2-\alpha)\phi - \phi^{n+1}\right], h_1 \\ &= -cy, h_2 = c^2\left[(k^2-\alpha)\phi - \phi^{n+1}\right]. \end{aligned} \tag{30}$$

Thus, we have

$$\begin{aligned} g(\phi,y,\tau) &= \tau cy - 2c^2\tau^2\left[(k^2-\alpha)\phi - \phi^{n+1}\right] + O(\tau^3), \\ h(\phi,y,\tau) &= -\tau cy + c^2\tau^2\left[(k^2-\alpha)\phi - \phi^{n+1}\right] + O(\tau^3). \end{aligned} \tag{31}$$

Precisely, the slow system (22) restricted to M_ε is

$$\begin{cases} \phi' = y, \\ y' = (k^2-\alpha)\phi - \phi^n(\varphi+g) - 2\gamma\phi^2 y. \end{cases} \tag{32}$$

Obviously, when $\tau=0$, system (32) reduces to (23). The equilibria $E_{\tau 0}$ and $E_{\tau 1}$ of system (32) are near to E_0 and E_1, respectively. In order to prove the existence of traveling wave fronts of equation (4), we establish the existence of a heteroclinic orbit connecting the critical point $E_{\tau 0}$. From Lemma 1, we know that such a heteroclinic orbit exists when $\tau = 0$.

Denote that (ϕ_1, y_1) and $(\bar{u}_0, \bar{v}_0)$ are the solutions of (10) and (32), respectively, when $\tau = 0$. For $\tau > 0$, set that

$$\phi_1 = \bar{u}_0 + \tau\widetilde{\phi} + O(\tau^2), y_1 = \bar{v}_0 + \tau\tilde{y} + O(\tau^2). \tag{33}$$

Substituting ϕ_1 and y_1 in (33) into (32) and comparing the coefficients of τ with each degree, the differential equation system determining $\widetilde{\phi}$ and $\tilde{y}$ is

$$\frac{d}{d\xi}\begin{pmatrix} \widetilde{\phi} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ c + \alpha + \beta\bar{u}_0 + \gamma\bar{u}_0^2 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\phi} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \beta c\bar{v}_0 \end{pmatrix}. \tag{34}$$

We intend to find a traveling wave solution satisfying (34) satisfying $\phi^{\sim}(\pm\infty) = 0, y^{\sim}(\pm\infty) = 0$. Denote that L^2 is the space of square integrable functions with inner production,

$$\left\langle \widetilde{\phi}(\xi), \tilde{y}(\xi) \right\rangle = \int_{-\infty}^{+\infty} \left(\widetilde{\phi}(\xi), \tilde{y}(\xi)\right) d\xi, \tag{35}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on R^2. From Fredholm theory, system (34) has a solution if and only if

$$\int_{-\infty}^{+\infty} \widetilde{\phi}(\xi), \left(\begin{pmatrix} 0 \\ \beta c\bar{v}_0 \end{pmatrix}\right) d\xi = 0 \tag{36}$$

holds for all functions $\widetilde{\phi}(\xi)$ in the kernel of the adjoint of operator L defined by the left-hand side of (34). Denote that L^* is the adjoint of operator L, and

$$L^* = -\frac{d}{d\xi} + \begin{pmatrix} 0 & c + \alpha + \beta\bar{u}_0 + \gamma\bar{u}_0^2 \\ -1 & 0 \end{pmatrix}. \tag{37}$$

It implies that for all $u_1(\xi) \in \text{Ker}, L^*$ satisfy

$$\frac{d\widetilde{\phi}(\xi)}{d\xi} = \begin{pmatrix} 0 & c + \alpha + \beta\bar{u}_0 + \gamma\bar{u}_0^2 \\ -1 & 0 \end{pmatrix}\widetilde{\phi}(\xi). \tag{38}$$

Since the matrix in (38) is a variable coefficient matrix, it is difficult to find the general solution. However, we can prove that such solution satisfying $u_0(-\infty) = 0$ must be the zero solution to deduce the existence of heteroclinic orbit. Although we can not find the exact expression, but $u_0(\xi)$ is the solutions for unperturbed system and $u_0(-\infty) = 0$. Thus, when $\xi \longrightarrow -\infty$, the matrix in (38) becomes a constant coefficient matrix

$$\begin{pmatrix} 0 & c + \alpha \\ -1 & 0 \end{pmatrix}. \tag{39}$$

The corresponding eigenvalues are determined by $\lambda^2 + c + \alpha = 0$. Since $c > 0, \alpha > 0$, there are two real nonzero eigenvalues $\lambda = \pm\sqrt{c + \alpha}i$. Hence, the solution satisfying $\phi^{\sim}(\pm\infty) = 0$ is the zero solution, which means that the Fredholm orthogonality condition holds trivially and so solutions of (38) exist, which satisfy $\phi^{\sim}(\pm\infty) = 0$ and $y^{\sim}(\pm\infty) = 0$. Therefore, we can conclude that for sufficiently small $\tau > 0$, there exists a heteroclinic orbit of (38) connecting $E_{0\tau}$ which is approaching to $E_0(0, 0)$ as $\tau \longrightarrow 0$.

Theorem 4. *Assume that $k^2 - \alpha > 1$, $\gamma > 2$, and $1 \leq n < 4$, (4) with the local strong generic kernel*

$$(f * u)(x, t) = \int_{-\infty}^{t} \frac{t - s}{\tau^2} e^{-((t-s)/\tau)} u(x, t) ds \tag{40}$$

possesses a traveling wave front $u(x, t) = u(x + ct)$ satisfying $u(-\infty) = 0, u(+\infty) = \sqrt[n]{k^2 - \alpha}$ when the parameter $\tau > 0$ is sufficiently small.

Remark 5. The time delay term $f * u$ in equation (4) is local strong generic delay convolution, which can be replaced by other delay convolution; the approach is still applicable provided the persistence of the traveling wave fronts, so we only consider the strong generic delay convolution kernel to investigate the traveling wave fronts for equation (4).

4. Conclusion

The paper investigates the existence of traveling wave fronts for a generalized nonlinear Schrodinger equation without delay, with a special local strong generic delay convolution kernel, respectively. Based on the relation between traveling wave fronts and heteroclinic orbit of the associated ordinary differential equations, by applying geometric singular perturbation theory, the singular perturbation system is changed to the regular perturbation system. Then, the Fredholm theory and linear chain trick are used to prove that the solution is traveling wave front solution on certain parametric conditions. The sufficient conditions of the traveling wave fronts that persisted for the generalized nonlinear Schrodinger equation (4) are given.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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