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Research Article

Analysis of Fractional Differential Equations with the Help of Different Operators

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This study uses an Elzaki decomposition method with two fractional derivatives to solve a fractional nonlinear coupled system of Whitham-Broer-Kaup equations. For the fractional derivatives, we used Caputo and Atangana-Baleanu derivatives in the Caputo manner. Furthermore, the proposed techniques are compared to the solutions of other renowned analytical methods, including the Adomian decomposition technique, variation iteration technique, and homotopy perturbation technique. We used two nonlinear problems to illustrate the accuracy and validity of the proposed approaches. The results of numerical simulations were used to verify that the proposed methods are accurate and efficient, and the results are displayed in graphs and tables. The obtained results demonstrate that the algorithm is very real, simple to apply, and effective in investigating the nature of complicated nonlinear models in science and engineering.

1. Introduction

In 1695, Leibniz presented fractional calculus (FC), one of the advancements of standard calculus [1]. In recent decades, the FC theory has played a significant role in physics, entropy, fluid mechanics, and engineering [2–5]. Using fractional calculus, specific physical models and engineering processes can be explained more precisely and practically. For instance, entropies based on fractional calculus may be applied more generally than Shannon entropy [6]. Due to its vast application, fractional entropy has been a popular subject of study [7]. Furthermore, fractional differential equations are effective for modelling several events [8]. This is because the next state of a system is decided not just by its current form, but also by all of its prior conditions. Such equations may mimic physical reality more closely than integer-order differential equations. It is important to note that the theory and applications of fractional calculus have been thoroughly studied in the literature [9–13].

Due to the accurate description of complicated events in system identification, non-Brownian motion, control problems, viscoelastic materials, polymers, and signal processing, fractional differential equations (FDEs) have garnered considerable attention in recent decades [14]. FDEs are nonlocal, which means that the next state of a system is determined not just by its current state but also by all of its prior states [15]. Using fractional derivatives, the fluid-dynamic traffic model, for instance, can overcome the weakness caused by the assumption of continuous traffic flow [15, 16]. Recent research has focused on fractional functional analysis [17, 18]. The characteristics and theorems of Yang-Fourier and Yang-Laplace transform, as well as their applications to fractional ordinary differential equations, fractional ordinary differential systems, and fractional partial differential equations, have been investigated.

The logical question is "How can we find the exact solutions to FDEs?" To comprehend the mechanics of complex nonlinear physical phenomena and implement them in daily life, nonlinear fractional differential equations (FDEs) have an important role in studying various areas of engineering, physics, and applied mathematics. In order to obtain numerical and analytical solutions of PDEs, a number of potent techniques, such as the Elzaki transform decomposition method [19, 20], the Iterative Laplace transform method [21], the variational iteration method [22], the Laplace transform decomposition method [23], the differential transform method [24], and the homotopy perturbation method [25], have been numerous scholars that have researched and solved numerous FDEs, including impulsive fractional differential equations [26], space and time-fractional advection-dispersion equation [27], and fractional generalised Burgers fluid [28].

Many well-known integral models, such as the KdV equation, Boussinesq equation, K-P equation, and WBK equation, are used to represent the propagation of shallow water. Whitham, Broer, and Kaup [29–31] developed nonlinear WBK equations using the Boussinesq approximation:

$$\begin{aligned} \mathbb{J}_{\rho} + \mathbb{J}\mathbb{J}_{\zeta} + \mathbb{K}_{\zeta} + q\mathbb{J}_{\zeta\zeta} &= 0, \\ \mathbb{K}_{\rho} + \mathbb{K}\mathbb{J}_{\zeta} + \mathbb{J}\mathbb{K}_{\zeta} - q\mathbb{K}_{\zeta\zeta} + p\mathbb{J}_{\zeta\zeta\zeta} &= 0, \end{aligned} \tag{1}$$

where $\mathbb{J} = \mathbb{J}(\zeta, \rho)$, $\mathbb{K} = \mathbb{K}(\zeta, \rho)$ denotes the horizontal velocity and height of the fluids, which fluctuate substantially from equilibrium, and q, p are constants made up of various diffusion powers. Wang and Zheng [32] employed an extended fractional Riccati subequation approach to get approximate solutions for the coupled system of (WBK) equations for fractional order (2). El-Borai et al. [33] used the exponential function method to solve coupled system (2). Author [34] employed the coupled fractional reduced differential transform method (CFRDTM) to get approximate analytical solutions to the model as mentioned earlier (2). The authors of [2] investigated numerical solutions to the specified coupled system using the residual power series method (RPSM) (2). Also employed to obtain numerical solutions to the coupled system (2) are the finite element method [36], the finite difference approach [35], the exponential-function method [37], variation iteration method (VIM), homotopy perturbation method (HPM), homotopy analysis method (HAM), and others [38–40].

Adomian introduced the Adomian decomposition methodology (ADM) in 1980, which is a method for locating numerical and explicit solutions to various differential equations that represent physical conditions. This method is applicable to initial value problems, boundary value problems, partial and ordinary differential equations, including linear and nonlinear equations, and stochastic systems. Combining the Adomian decomposition method and the Elzaki transform method yields the Elzaki transform decomposition method (ETDM). The ETDM has also been utilized in several studies to solve fractional-order nonlinear partial differential equations numerically [41, 42].

In 1998, He was the first to introduce the homotopy perturbation method (HPM) [43, 44]. Later on, the solutions of some nonlinear nonhomogeneous partial differential equations are obtained through this semianalytical method [45, 46]. The solution that they get is in the form of an infinite sequence that converges rapidly to the exact solutions. Due to its quick results, the method was further used for solving linear and nonlinear equations. In the present work, we used an approximate analytical technique that combines the Elzaki transform and HPM, known as the HPTM. The proposed methods and solutions are in good agreement with the exact solution of the targeted problems. The fractional view analysis of the problems is also shown using the suggested techniques. It is noticed that the proposed methods can be modified to solve other fractional PDEs and their systems [47, 48]. In this study, we apply ETDM with two different derivatives to investigate the general and numerical solution of the coupled system of fractional-order Whitham-Broer-Kaup equations, as suggested by the studies mentioned above. ETDM is a straightforward and effective technique that requires no disturbance. We compare the outcomes of our proposed method to those of well-known methodologies such as VIM, ADM, and OHAM. We may observe that the provided strategy for finding solutions to nonlinear fractional-order partial differential equations is superior to the previously discussed method. We execute the calculations with Maple. The convergence of the proposed method is also ensured by extending the concept described in [49, 50].

2. Basic Definitions

This section introduces the essential ideas of fractional derivatives, fractional integrals, and the Elzaki transform with and without a singular kernel.

Definition 1. The fractional Caputo derivative (CFD) is given as follows:

$$D^{\delta}_{\wp}(\ell(\wp)) = \begin{cases} \dfrac{1}{\Gamma(m-\delta)} \displaystyle\int_0^{\wp} \dfrac{\ell^m(\eta)}{(\wp-\eta)^{\delta+1-m}} d\eta, & m-1 < \delta < m, \\ \dfrac{d^m}{d\wp^m} \ell(\wp), & \delta = m. \end{cases} \tag{2}$$

Definition 2. The derivative in terms of the Atangana-Baleanu Caputo manner (ABC) is given as follows:

$$D^{\delta}_{\wp}(\ell(\wp)) = \frac{N(\delta)}{1-\delta} \int_m^{\wp} \ell'(\eta) E_{\delta}\left[-\frac{\delta(\wp-\eta)^{\delta}}{1-\delta}\right] d\eta, \tag{3}$$

where $\ell \in H^1(\alpha, \beta), \beta > \alpha, \delta \in [0, 1]$. A normalisation function equal to 1 when $\delta = 0$ and $\delta = 1$ is represented by $N(\delta)$ in equation (11).

Definition 3. The ABC fractional integral operator is as follows:

$$I^{\delta}_{\wp}(\ell(\wp)) = \frac{1-\delta}{N(\delta)} \ell(\wp) + \frac{\delta}{\Gamma(\delta) N(\delta)} \int_m^{\wp} \ell(\eta)(\wp-\eta)^{\delta-1} d\eta. \tag{4}$$

Definition 4. The Elzaki transform's exponential function is given as in set A:

$$A = \left\{\ell(\wp) : \exists G, p_1, p_2 > 0, |\ell(\wp)| < Ge^{|\wp|/p_j}, \quad \text{if}\, \wp \in (-1)^j \times [0,\infty)\right\}. \tag{5}$$

For a certain function in the set, G is a finite number but p_1, p_2 can be finite or infinite.

Definition 5. For the function $\ell(\wp)$, the transformation in terms of Elzaki is as follows:

$$\mathscr{E}\{\ell(\wp)\}(\varpi) = \tilde{U}(\varpi) = \varpi \int_0^\infty e^{-(\wp/\varpi)} \ell(\wp) d\wp, \tag{6}$$

where $\wp \geq 0, p_1 \leq \varpi \leq p_2$.

Theorem 6 (Elzaki transformation convolution theorem). *The following equality holds:*

$$\mathscr{E}\{\ell * \nu\} = \frac{1}{\varpi} \mathscr{E}(\ell) \mathscr{E}(\nu), \tag{7}$$

where Elzaki transform is indicated by $\mathscr{E}\{.\}$.

Definition 7. The Elzaki transform of the CFD operator $D_\wp^\delta (\ell(\wp))$ is given by

$$\mathscr{E}\left\{D_\wp^\delta (\ell(\wp))\right\}(\varpi) = \varpi^{-\delta} \tilde{U}(\varpi) - \sum_{k=0}^{m-1} \varpi^{2-\delta+k} \ell^k(0), \tag{8}$$

where $m-1 < \delta < m$.

Theorem 8. *The ABC fractional derivative* $D_\wp^\delta (\ell(\wp))$ *Elzaki transform is defined as follows:*

$$\mathscr{E}\left\{D_\wp^\delta (\ell(\wp))\right\}(\varpi) = \frac{N(\delta)\varpi}{\delta\varpi^\delta + 1 - \delta} \left(\frac{\tilde{U}(\varpi)}{\varpi} - \varpi \ell(0)\right), \tag{9}$$

where $\mathscr{E}\{\ell(\wp)\}\varpi = \tilde{U}(\varpi)$.

Proof. From Definition 2, we have the following:

$$\mathscr{E}\left\{D_\wp^\delta (\ell(\wp))\right\}(\varpi) = \mathscr{E}\left\{\frac{N(\delta)}{1-\delta} \int_0^\wp \ell'(\eta) E_\delta \left[-\frac{\delta(\wp-\eta)^\delta}{1-\delta}\right] d\eta\right\}(\varpi). \tag{10}$$

□

Then, taking into account the definition and convolution of the Elzaki transform, we get the following:

$$\begin{aligned}
&\mathscr{E}\left\{D_\wp^\delta (\ell(\wp))\right\}(\varpi) \\
&= \mathscr{E}\left\{\frac{N(\delta)}{1-\delta} \int_0^\wp \ell'(\eta) E_\delta \left[-\frac{\delta(\wp-\eta)^\delta}{1-\delta}\right] d\eta\right\} \\
&= \frac{N(\delta)}{1-\delta} \frac{1}{\varpi} \mathscr{E}\left\{\ell'(\eta)\right\} \mathscr{E}\left\{E_\delta \left[-\frac{\delta\wp^\delta}{1-\delta}\right] d\eta\right\} \\
&= \frac{N(\delta)}{1-\delta} \left[\frac{\tilde{U}(\varpi)}{\varpi} - \varpi\ell(0)\right] \left[\int_0^\infty e^{-(1/\varpi)} E_\delta \left[-\frac{\delta\wp^\delta}{1-\delta}\right] d\wp\right] \\
&= \frac{N(\delta)\varpi}{\delta\varpi^\delta + 1 - \delta} \left[\frac{\tilde{U}(\varpi)}{\varpi} - \varpi\ell(0)\right].
\end{aligned} \tag{11}$$

3. Methodology

Here, we give the general methodology of the proposed technique to solve the given equation.

$$D_\rho^\delta \mathbb{J}(\zeta,\rho) = \mathscr{L}(\mathbb{J}(\zeta,\rho)) + N(\mathbb{J}(\zeta,\rho)) + h(\zeta,\rho) = M(\zeta,\rho), \tag{12}$$

with initial condition

$$\mathbb{J}(\zeta, 0) = \phi(\zeta), \tag{13}$$

having $\mathscr{L}$, N linear and nonlinear terms and $h(\zeta,\rho)$ is the source term.

3.1. Case I ($ETDM_C$). By means of Caputo fractional derivative and Elzaki transform, equation (12) can be stated as follows:

$$\frac{1}{p(\delta,\ell,\kappa)} \left(\mathbf{E}[\mathbb{J}(\zeta,\rho)] - \kappa^2\phi(\zeta)\right) = \mathbf{E}[M(\zeta,\rho)], \tag{14}$$

with

$$p(\delta,\ell,\kappa) = \kappa^\delta. \tag{15}$$

On employing the Elzaki inverse transform, we have

$$\mathbb{J}(\zeta,\rho) = \mathbf{E}^{-1}\left(\kappa^2\phi(\zeta) + p(\delta,\ell,\kappa)\mathbf{E}[M(\zeta,\rho)]\right). \tag{16}$$

Thus, for $\mathbb{J}(\zeta,\rho)$, the solution in the series form is stated as follows:

$$\mathbb{J}(\zeta,\rho) = \sum_{i=0}^\infty \mathbb{J}_i(\zeta,\rho). \tag{17}$$

And $N(\mathbb{J}(\zeta,\rho))$ can be decomposed as follows:

$$N(\mathbb{J}(\zeta,\rho)) = \sum_{i=0}^{\infty} A_i(\mathbb{J}_0,\cdots,\mathbb{J}_i), \tag{18}$$

having A_i as the Adomian polynomials and can be calculated as

$$A_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}N\rho, \quad \Sigma_{k=0}^{n}\varepsilon^k \mathbb{J}_k\Big|_{\varepsilon=0}. \tag{19}$$

Putting equations (18) and (17) into (16), we obtain

$$\sum_{i=0}^{\infty}\mathbb{J}_i(\zeta,\rho) = \mathbf{E}^{-1}\left(\kappa^2\phi(\zeta) + p(\delta,\ell,\kappa)\mathbf{E}[h(\zeta,\rho)]\right) + \mathbf{E}^{-1}\left(p(\delta,\ell,\kappa)\mathbf{E}\left[\sum_{i=0}^{\infty}\mathscr{L}(\mathbb{J}_i(\zeta,\rho)) + A_\rho\right]\right). \tag{20}$$

From (20), we get

$$\begin{aligned}
\mathbb{J}_0^C(\zeta,\rho) &= \mathbf{E}^{-1}\left(\frac{\phi(\zeta)}{\kappa} + p(\delta,\ell,\kappa)\mathbf{E}[h(\zeta,\rho)]\right),\\
\mathbb{J}_1^C(\zeta,\rho) &= \mathbf{E}^{-1}(p(\delta,\ell,\kappa)\mathbf{E}[\mathscr{L}(\mathbb{J}_0(\zeta,\rho)) + A_0]),\\
&\vdots\\
\mathbb{J}_{l+1}^C(\zeta,\rho) &= \mathbf{E}^{-1}(p(\delta,\ell,\kappa)\mathbf{E}[\mathscr{L}(\mathbb{J}_l(\zeta,\rho)) + A_l]), \quad l = 1,2,3\cdots.
\end{aligned} \tag{21}$$

Thus, we get the solution of (12) by substituting (21) into (17) using $ETDM_C$:

$$\mathbb{J}^C(\zeta,\rho) = \mathbb{J}_0^C(\zeta,\rho) + \mathbb{J}_1^C(\zeta,\rho) + \mathbb{J}_2^C(\zeta,\rho) + \cdots. \tag{22}$$

3.2. Case II ($ETDM_{ABC}$). By means of ABC fractional derivative and Elzaki transform, equation (12) can be stated as follows:

$$\frac{1}{q(\delta,\ell,\kappa)}\left(\mathbf{E}[\mathbb{J}(\zeta,\rho)] - \frac{\phi(\zeta)}{\kappa}\right) = \mathbf{E}[M(\zeta,\rho)], \tag{23}$$

with

$$q(\delta,\ell,\kappa) = \frac{1-\delta+\delta(\ell/\kappa)^\delta}{B(\delta)}. \tag{24}$$

On employing the Elzaki inverse transform, we have

$$\mathbb{J}(\zeta,\rho) = \mathbf{E}^{-1}\left(\frac{\phi(\zeta)}{\kappa} + q(\delta,\ell,\kappa)\mathbf{E}[M(\zeta,\rho)]\right). \tag{25}$$

By means of Adomian decomposition, we get

$$\sum_{i=0}^{\infty}\mathbb{J}_i(\zeta,\rho) = \mathbf{E}^{-1}\left(\frac{\phi(\zeta)}{\kappa} + q(\delta,\ell,\kappa)\mathbf{E}[h(\zeta,\rho)]\right) + \mathbf{E}^{-1}\left(q(\delta,\ell,\kappa)\mathbf{E}\left[\sum_{i=0}^{\infty}\mathscr{L}(\mathbb{J}_i(\zeta,\rho)) + A_\rho\right]\right). \tag{26}$$

From (20), we get

$$\begin{aligned}
\mathbb{J}_0^{ABC}(\zeta,\rho) &= \mathbf{E}^{-1}\left(\frac{\phi(\zeta)}{\kappa} + q(\delta,\ell,\kappa)\mathbf{E}[h(\zeta,\rho)]\right),\\
\mathbb{J}_1^{ABC}(\zeta,\rho) &= \mathbf{E}^{-1}(q(\delta,\ell,\kappa)\mathbf{E}[\mathscr{L}(\mathbb{J}_0(\zeta,\rho)) + A_0]),\\
&\vdots\\
\mathbb{J}_{l+1}^{ABC}(\zeta,\rho) &= \mathbf{E}^{-1}(q(\delta,\ell,\kappa)\mathbf{E}[\mathscr{L}(\mathbb{J}_l(\zeta,\rho)) + A_l]), l = 1,2,3\cdots.
\end{aligned} \tag{27}$$

Thus, we get the solution of (12), by using $ETDM_{ABC}$

$$\mathbb{J}^{ABC}(\zeta,\rho) = \mathbb{J}_0^{ABC}(\zeta,\rho) + \mathbb{J}_1^{ABC}(\zeta,\rho) + \mathbb{J}_2^{ABC}(\zeta,\rho) + \cdots. \tag{28}$$

4. Applications

In this part, we implemented the proposed technique to solve nonlinear systems of Whitham-Broer-Kaup equations having order fraction.

Example 9. Let us consider the fractional WBKEs system:

$$\begin{aligned}
&D_\rho^\delta\mathbb{J}(\zeta,\rho) + \mathbb{J}(\zeta,\rho)\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta} = 0,\\
&D_\rho^\delta\mathbb{K}(\zeta,\rho) + \mathbb{J}(\zeta,\rho)\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta} + 3\frac{\partial^3\mathbb{J}(\zeta,\rho)}{\partial\zeta^3} - \frac{\partial^2\mathbb{K}(\zeta,\rho)}{\partial\zeta^2} = 0,\\
&0 < \delta \le 1, -1 < \rho \le 1, \quad -10 \le \zeta \le 10,
\end{aligned} \tag{29}$$

having initial condition

$$\begin{aligned}
\mathbb{J}(\zeta,0) &= \frac{1}{2} - 8\tan h(-2\zeta),\\
\mathbb{K}(\zeta,0) &= 16 - 16\tan h^2(-2\zeta).
\end{aligned} \tag{30}$$

On employing the Elzaki transform, we have

$$\begin{aligned}
\mathbf{E}\left[D_\rho^\delta \mathbb{J}(\zeta,\rho)\right] &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta}\right],\\
\mathbf{E}\left[D_\rho^\delta \mathbb{K}(\zeta,\rho)\right] &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + 3\frac{\partial^3 \mathbb{J}(\zeta,\rho)}{\partial \zeta^3} - \frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial \zeta^2}\right],
\end{aligned} \tag{31}$$

Thus, we have

$$\begin{aligned}
\frac{1}{\kappa^\delta}\mathbf{E}[\mathbb{J}(\zeta,\rho)] - \kappa^{2-\delta}\mathbb{J}(\zeta,0) &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta}\right],\\
\frac{1}{\kappa^\delta}\mathbf{E}[\mathbb{K}(\zeta,\rho)] - \kappa^{2-\delta}\mathbb{J}(\zeta,0) &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + 3\frac{\partial^3 \mathbb{J}(\zeta,\rho)}{\partial \zeta^3} - \frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial \zeta^2}\right].
\end{aligned} \tag{32}$$

On simplification, we have

$$\begin{aligned}
\mathbf{E}[\mathbb{J}(\zeta,\rho)] &= \kappa^2\left[\frac{1}{2} - 8\tan h(-2\zeta)\right] - \kappa^\delta \mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta}\right],\\
\mathbf{E}[\mathbb{K}(\zeta,\rho)] &= \kappa^2\left[16 - 16\tan h^2(-2\zeta)\right] - \kappa^\delta \mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + 3\frac{\partial^3 \mathbb{J}(\zeta,\rho)}{\partial \zeta^3} - \frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial \zeta^2}\right].
\end{aligned} \tag{33}$$

On applying the inverse ET, we get

$$\begin{aligned}
\mathbb{J}(\zeta,\rho) &= \left[\frac{1}{2} - 8\tan h(-2\zeta)\right] - \mathbf{E}^{-1}\left[\kappa^\delta \mathbf{E}\left\{\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta}\right\}\right],\\
\mathbb{K}(\zeta,\rho) &= \left[16 - 16\tan h^2(-2\zeta)\right] - \mathbf{E}^{-1}\left[\kappa^\delta \mathbf{E}\left\{\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + 3\frac{\partial^3 \mathbb{J}(\zeta,\rho)}{\partial \zeta^3} - \frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial \zeta^2}\right\}\right].
\end{aligned} \tag{34}$$

4.1. Solution by Means of EDM_C. The solutions in the series form for the unknown function $\mathbb{J}(\zeta,\rho)$ and $\mathbb{K}(\zeta,\rho)$ are stated as follows:

$$\mathbb{J}(\zeta,\rho) = \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho),$$

$$\mathbb{K}(\zeta,\rho) = \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho). \tag{35}$$

The nonlinear terms by means of Adomian polynomials are stated as $\mathbb{J}\mathbb{J}_\zeta = \sum_{m=0}^{\infty}\mathcal{A}_m$, $\mathbb{J}\mathbb{K}_\zeta = \sum_{m=0}^{\infty}\mathcal{B}_m$ and $\mathbb{K}\mathbb{J}_\zeta = \sum_{m=0}^{\infty}\mathcal{C}_m$; thus, by means of these terms, equation (34) can be determined as follows:

$$\begin{aligned}
\sum_{l=0}^{\infty}\mathbb{J}_{l+1}(\zeta,\rho) &= \frac{1}{2} - 8\tan h(-2\zeta) - \mathbf{E}^{-1}\left[\kappa^\delta \mathbf{E}\left\{\sum_{l=0}^{\infty}\mathcal{A}_l + \frac{\partial \mathbb{J}(\zeta,\rho)}{\partial \zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial \zeta}\right\}\right],\\
\sum_{l=0}^{\infty}\mathbb{K}_{l+1}(\zeta,\rho) &= 16 - 16\tan h^2(-2\zeta) - \mathbf{E}^{-1}\left[\kappa^\delta \mathbf{E}\left\{\sum_{l=0}^{\infty}\mathcal{B}_l + \sum_{l=0}^{\infty}\mathcal{C}_l + 3\frac{\partial^3 \mathbb{J}(\zeta,\rho)}{\partial \zeta^3} - \frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial \zeta^2}\right\}\right].
\end{aligned} \tag{36}$$

By the comparison of both sides of equation (36), we obtain

$$\begin{aligned}
\mathbb{J}_0(\zeta,\rho) &= \frac{1}{2} - 8\tan h(-2\zeta),\\
\mathbb{K}_0(\zeta,\rho) &= 16 - 16\tan h^2(-2\zeta),\\
\mathbb{J}_1(\zeta,\rho) &= -8\sec h^2(-2\zeta)\frac{\rho^\delta}{\Gamma(\delta+1)},\\
\mathbb{K}_1(\zeta,\rho) &= -32\sec h^2(-2\zeta)\tan h(-2\zeta)\frac{\rho^\delta}{\Gamma(\delta+1)},\\
\mathbb{J}_2(\zeta,\rho) &= -16\sec h^2(-2\zeta)\big(4\sec h^2(-2\zeta) - 8\tan h^2(-2\zeta)\\
&\quad + 3\tan h(-2\zeta)\big)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)},\\
\mathbb{K}_2(\zeta,\rho) &= -32\sec h^2 2(-2\zeta)\big\{40\sec h^2(-2\zeta)\tan h(-2\zeta)\\
&\quad + 96\tan h(-2\zeta) - 2\tan h^2(-2\zeta)\\
&\quad - 32\tan h^3(-2\zeta) - 25\sec h^2(-2\zeta)\big\}\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}.
\end{aligned} \tag{37}$$

Thus, for $\mathbb{J}_l$ and $\mathbb{K}_l$ with $(l \geq 3)$, the remaining components are easily computable. So, the solution in series form is as follows:

$$\begin{aligned}
\mathbb{J}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho) = \mathbb{J}_0(\zeta,\rho) + \mathbb{J}_1(\zeta,\rho) + \mathbb{J}_2(\zeta,\rho) + \cdots,\\
\mathbb{J}(\zeta,\rho) &= \frac{1}{2} - 8\tan h(-2\zeta) - 8\sec h^2(-2\zeta)\frac{\rho^\delta}{\Gamma(\delta+1)}\\
&\quad - 16\sec h^2(-2\zeta)\big(4\sec h^2(-2\zeta) - 8\tan h^2(-2\zeta)\\
&\quad + 3\tan h(-2\zeta)\big)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)} + \cdots.\\
\mathbb{K}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{K}_l(\zeta,\rho) = \mathbb{K}_0(\zeta,\rho) + \mathbb{K}_1(\zeta,\rho) + \mathbb{K}_2(\zeta,\rho) + \cdots,\\
\mathbb{K}(\zeta,\rho) &= 16 - 16\tan h^2(-2\zeta) - 32\sec h^2(-2\zeta)\tan h(-2\zeta)\\
&\quad \cdot \frac{\rho^\delta}{\Gamma(\delta+1)} - 32\sec h^2(-2\zeta)\big\{40\sec h^2(-2\zeta)\tan h(-2\zeta)\\
&\quad + 96\tan h(-2\zeta) - 2\tan h^2(-2\zeta) - 32\tan h^3(-2\zeta)\\
&\quad - 25\sec h^2(-2\zeta)\big\}\frac{\rho^{2\delta}}{\Gamma(2\delta+1)} + \cdots.
\end{aligned} \tag{38}$$

4.2. Solution by Means of EDM_{ABC}. The solutions in the series form for the unknown function $\mathbb{J}(\zeta,\rho)$ and $\mathbb{K}(\zeta,\rho)$ are stated as follows:

$$\begin{aligned}
\mathbb{J}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho),\\
\mathbb{K}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho).
\end{aligned} \tag{39}$$

The nonlinear terms by means of Adomian polynomials are stated as $\mathbb{J}\mathbb{J}_\zeta = \sum_{l=0}^{\infty}\mathscr{A}_l$ and $\mathbb{J}^2\mathbb{J}_\zeta = \sum_{l=0}^{\infty}\mathscr{B}_l$; thus, by means of these terms, equation (34) can be determined as follows:

$$\sum_{l=0}^{\infty}\mathbb{J}_{l+1}(\zeta,\rho) = \frac{1}{2} - 8\tan h(-2\zeta) - \mathbf{E}^{-1}\left[\frac{\ell^\delta\left(\kappa^\delta + \delta\left(\ell^\delta - \kappa^\delta\right)\right)}{\kappa^{2\delta}}\mathbf{E}\left\{\sum_{l=0}^{\infty}\mathscr{A}_l + \frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}\right\}\right], \tag{40}$$

$$\sum_{l=0}^{\infty}\mathbb{K}_{l+1}(\zeta,\rho) = 16 - 16\tan h^2(-2\zeta) - \mathbf{E}^{-1}\left[\frac{\ell^\delta\left(\kappa^\delta + \delta\left(\ell^\delta - \kappa^\delta\right)\right)}{\kappa^{2\delta}}\mathbf{E}\left\{\sum_{l=0}^{\infty}\mathscr{B}_l + \sum_{l=0}^{\infty}\mathscr{C}_l + 3\frac{\partial^3\mathbb{J}(\zeta,\rho)}{\partial\zeta^3} - \frac{\partial^2\mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right\}\right]. \tag{41}$$

By the comparison of both sides of equation (41), we obtain

$$\begin{aligned}
\mathbb{J}_0(\zeta,\rho) &= \frac{1}{2} - 8\tan h(-2\zeta),\\
\mathbb{K}_0(\zeta,\rho) &= 16 - 16\tan h^2(-2\zeta),\\
\mathbb{J}_1(\zeta,\rho) &= -8\sec h^2(-2\zeta)\left(1 - \delta + \frac{\delta\rho^\delta}{\Gamma(\delta+1)}\right),\\
\mathbb{K}_1(\zeta,\rho) &= -32\sec h^2(-2\zeta)\tan h(-2\zeta)\left(1 - \delta + \frac{\delta\rho^\delta}{\Gamma(\delta+1)}\right),\\
\mathbb{J}_2(\zeta,\rho) &= -16\sec h^2(-2\zeta)\big(4\sec h^2(-2\zeta)\\
&\quad - 8\tan h^2(-2\zeta) + 3\tan h(-2\zeta)\big)\\
&\quad \cdot\left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^\delta}{\Gamma(\delta+1)} + (1-\delta)^2\right],\\
\mathbb{K}_2(\zeta,\rho) &= -32\sec h^2(-2\zeta)\big\{40\sec h^2(-2\zeta)\tan h(-2\zeta)\\
&\quad + 96\tan h(-2\zeta) - 2\tan h^2(-2\zeta)
\end{aligned}$$

$$- 32 \tan h^3(-2\zeta) - 25 \sec h^2(-2\zeta)\} \cdot \left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^\delta}{\Gamma(\delta+1)} + (1-\delta)^2\right]. \tag{42}$$

Thus, for $\mathbb{J}_l$ with $(l \geq 3)$, the remaining components are easily computable. So, the solution in the series form is as follows:

$$\mathbb{J}(\zeta,\rho) = \sum_{l=0}^{\infty} \mathbb{J}_l(\zeta,\rho) = \mathbb{J}_0(\zeta,\rho) + \mathbb{J}_1(\zeta,\rho) + \mathbb{J}_2(\zeta,\rho) + \cdots,$$

$$\begin{aligned}\mathbb{J}(\zeta,\rho) = {} & \frac{1}{2} - 8 \tan h(-2\zeta) - 8 \sec h^2(-2\zeta)\left(1 - \delta + \frac{\delta\rho^\delta}{\Gamma(\delta+1)}\right) \\ & - 16 \sec h^2(-2\zeta)\big(4 \sec h^2(-2\zeta) \\ & - 8 \tan h^2(-2\zeta) + 3 \tan h(-2\zeta)\big) \\ & \cdot \left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^\delta}{\Gamma(\delta+1)} + (1-\delta)^2\right] + \cdots,\end{aligned}$$

$$\mathbb{K}(\zeta,\rho) = \sum_{l=0}^{\infty} \mathbb{K}_l(\zeta,\rho) = \mathbb{K}_0(\zeta,\rho) + \mathbb{K}_1(\zeta,\rho) + \mathbb{K}_2(\zeta,\rho) + \cdots,$$

$$\begin{aligned}\mathbb{K}(\zeta,\rho) = {} & 16 - 16 \tan h^2(-2\zeta) - 32 \sec h^2(-2\zeta) \tan h(-2\zeta) \\ & \cdot \left(1 - \delta + \frac{\delta\rho^\delta}{\Gamma(\delta+1)}\right) - 32 \sec h^2(-2\zeta) \\ & \cdot \{40 \sec h^2(-2\zeta) \tan h(-2\zeta) + 96 \tan h(-2\zeta) \\ & - 2 \tan h^2(-2\zeta) - 32 \tan h^3(-2\zeta) - 25 \sec h^2(-2\zeta)\} \\ & \cdot \left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^\delta}{\Gamma(\delta+1)} + (1-\delta)^2\right] + \cdots.\end{aligned} \tag{43}$$

On taking $\delta = 1$, we obtain the exact solution as follows:

$$\mathbb{J}(\zeta,\rho) = \frac{1}{2} - 8 \tan h\left\{-2\left(\zeta - \frac{\rho}{2}\right)\right\},$$

$$\mathbb{K}(\zeta,\rho) = 16 - 16 \tan h^2\left\{-2\left(\zeta - \frac{\rho}{2}\right)\right\}. \tag{44}$$

Figure 1 shows a graphical view of the exact and analytical solution for $\mathbb{J}(\zeta,\rho)$ at $\delta = 1$ of system 1. Figure 2 shows a graphical view of the analytical solution for $\mathbb{J}(\zeta,\rho)$ at $\delta = 0.8, 0.6$ of system 1, and Figure 3 shows that of the analytical solution at various values of δ for $\mathbb{J}(\zeta,\rho)$ of system 1. Figure 4 shows the absolute error graph of $\mathbb{J}(\zeta,\rho)$ of system 1. Similarly, Figures 5–7 show that the exact and analytical solution for $\mathbb{K}(\zeta,\rho)$ at $\delta = 1$ of system 1 and $\mathbb{K}(\zeta,\rho)$ at the different fractional order of $\delta = 0.8, 0.6$ of system 1. Tables 1 and 2 show that the different fractional order of δ of system 1.

Example 10. Let us consider the fractional WBKE system:

$$\begin{aligned}& D_\rho^\delta \mathbb{J}(\zeta,\rho) + \mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{1}{2}\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial\zeta} = 0, \\ & D_\rho^\delta \mathbb{K}(\zeta,\rho) + \mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial\zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} - \frac{1}{2}\frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial\zeta^2} = 0, \\ & 0 < \delta \leq 1, \quad -1 < \rho \leq 1, -10 \leq \zeta \leq 10,\end{aligned} \tag{45}$$

having the initial condition

$$\begin{aligned}\mathbb{J}(\zeta,0) &= \lambda - \kappa \cot h[\kappa(\zeta+\theta)], \\ \mathbb{K}(\zeta,0) &= -\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta+\theta)].\end{aligned} \tag{46}$$

On employing the Elzaki transform, we have

$$\begin{aligned}\mathbf{E}\left[D_\rho^\delta \mathbb{J}(\zeta,\rho)\right] &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{1}{2}\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial\zeta}\right], \\ \mathbf{E}\left[D_\rho^\delta \mathbb{K}(\zeta,\rho)\right] &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial\zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} - \frac{1}{2}\frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right].\end{aligned} \tag{47}$$

Thus, we have

$$\begin{aligned}\frac{1}{\kappa^\delta}\mathbf{E}[\mathbb{J}(\zeta,\rho)] - \kappa^{2-\delta}\mathbb{J}(\zeta,0) &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{1}{2}\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} + \frac{\partial \mathbb{K}(\zeta,\rho)}{\partial\zeta}\right], \\ \frac{1}{\kappa^\delta}\mathbf{E}[\mathbb{K}(\zeta,\rho)] - \kappa^{2-\delta}\mathbb{J}(\zeta,0) &= -\mathbf{E}\left[\mathbb{J}(\zeta,\rho)\frac{\partial \mathbb{K}(\zeta,\rho)}{\partial\zeta} + \mathbb{K}(\zeta,\rho)\frac{\partial \mathbb{J}(\zeta,\rho)}{\partial\zeta} - \frac{1}{2}\frac{\partial^2 \mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right].\end{aligned} \tag{48}$$

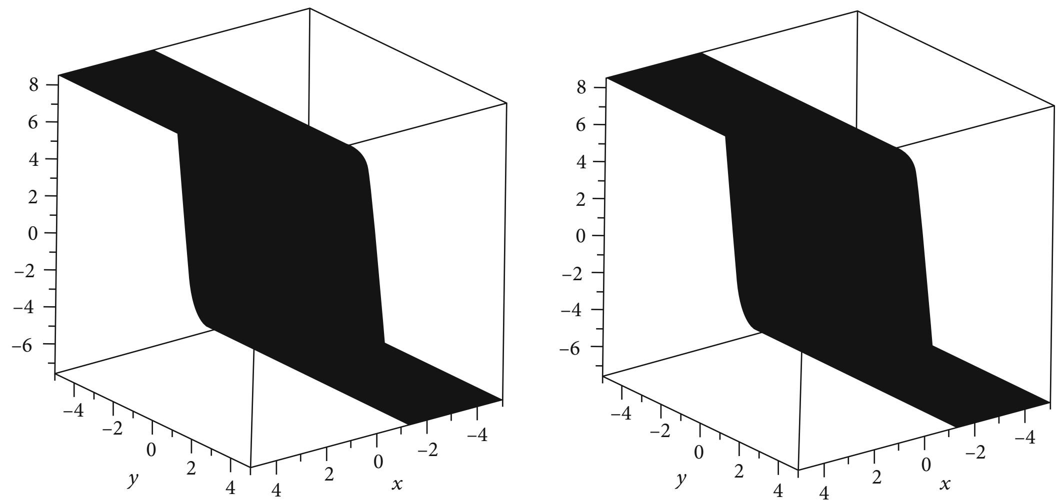

FIGURE 1: Graphical view of the exact and analytical solution for $\mathbb{J}(\zeta,\rho)$ at $\delta=1$ of system 1.

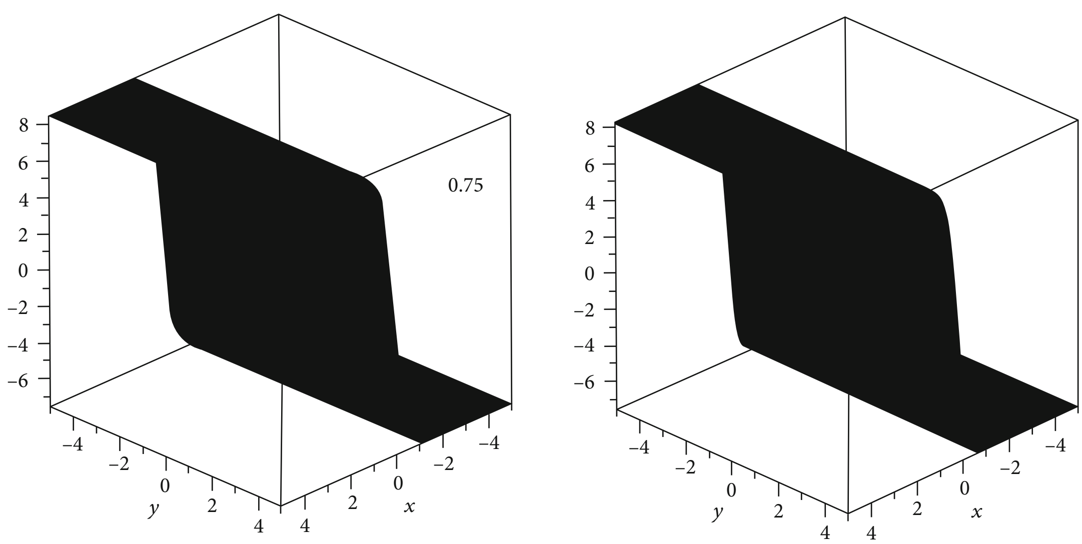

FIGURE 2: Graphical view of the analytical solution for $\mathbb{J}(\zeta,\rho)$ at $\delta=0.8,0.6$ of system 1.

On simplification, we have

$$\mathbf{E}[\mathbb{J}(\zeta,\rho)]=\kappa^2[\lambda-\kappa\cot h[\kappa(\zeta+\theta)]]-\kappa^\delta\mathbf{E}\cdot\left[\mathbb{J}(\zeta,\rho)\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}+\frac{1}{2}\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}+\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}\right],$$
$$\mathbf{E}[\mathbb{K}(\zeta,\rho)]=\kappa^2\left[-\kappa^2\operatorname{cosec} h^2[\kappa(\zeta+\theta)]\right]-\kappa^\delta\mathbf{E}\cdot\left[\mathbb{J}(\zeta,\rho)\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}+\mathbb{K}(\zeta,\rho)\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}-\frac{1}{2}\frac{\partial^2\mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right]. \tag{49}$$

On applying the inverse NT, we get

$$\mathbb{J}(\zeta,\rho)=\left[\frac{1}{2}-8\tan h(-2\zeta)\right]-\mathbf{E}^{-1}\cdot\left[\kappa^\delta\mathbf{E}\left\{\mathbb{J}(\zeta,\rho)\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}+\frac{1}{2}\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}+\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}\right\}\right],$$
$$\mathbb{K}(\zeta,\rho)=\left[16-16\tan h^2(-2\zeta)\right]-\mathbf{E}^{-1}\cdot\left[\kappa^\delta\mathbf{E}\left\{\mathbb{J}(\zeta,\rho)\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}+\mathbb{K}(\zeta,\rho)\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}-\frac{1}{2}\frac{\partial^2\mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right\}\right]. \tag{50}$$

4.3. Solution by Means of NDM_C. The solutions in series form for the unknown function $\mathbb{J}(\zeta,\rho)$ and $\mathbb{K}(\zeta,\rho)$ are stated as follows:

$$\mathbb{J}(\zeta,\rho)=\sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho)\text{ and }\mathbb{K}(\zeta,\rho)=\sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho). \tag{51}$$

The nonlinear terms by means of Adomian polynomials are stated as $\mathbb{J}\mathbb{J}_\zeta=\sum_{m=0}^{\infty}\mathscr{A}_m$, $\mathbb{J}\mathbb{K}_\zeta=\sum_{m=0}^{\infty}\mathscr{B}_m$ and $\mathbb{K}\mathbb{J}_\zeta=\sum_{m=0}^{\infty}\mathscr{C}_m$; thus, by means of these terms, equation (50) can be determined as follows:

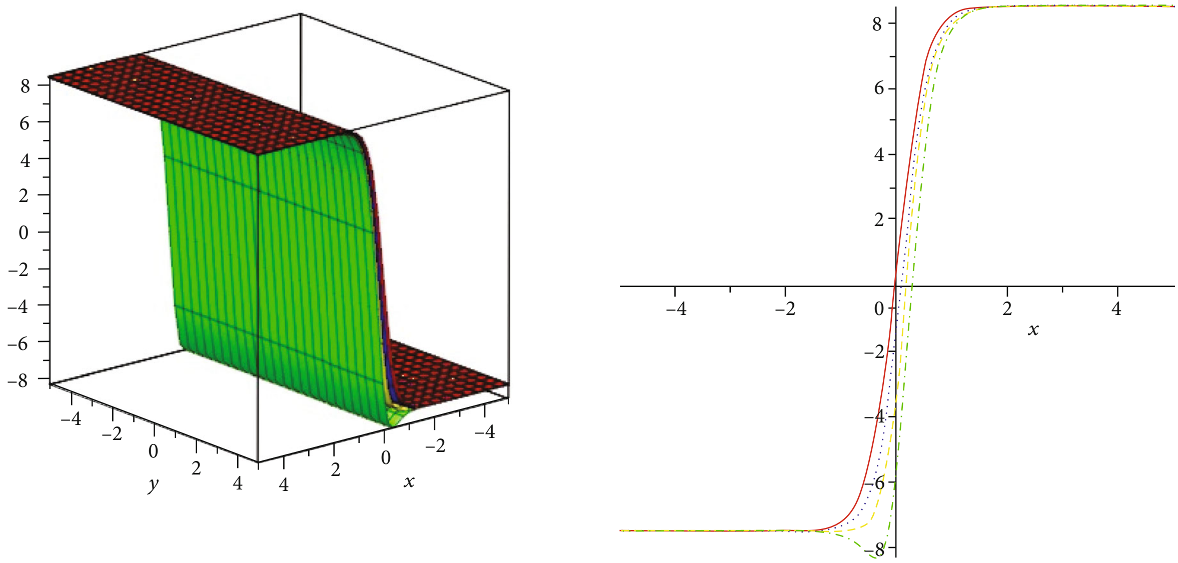

FIGURE 3: Graphical view of the analytical solution at various values of δ for $\mathbb{J}(\zeta,\rho)$ of system 1.

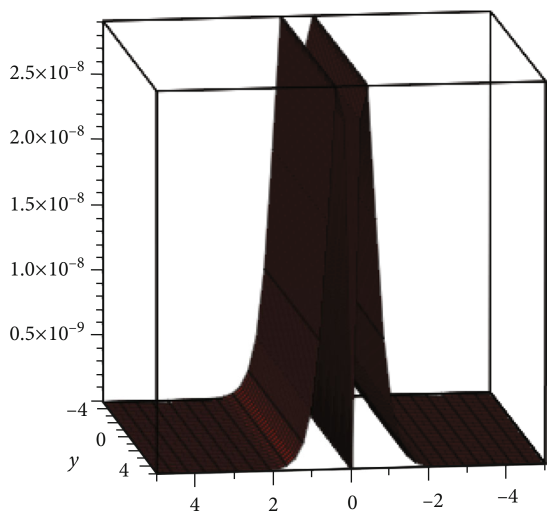

FIGURE 4: Graphical view of the absolute error for $\mathbb{J}(\zeta,\rho)$ of system 1.

$$\begin{aligned}\sum_{l=0}^{\infty}\mathbb{J}_{l+1}(\zeta,\rho)&=\lambda-\kappa\cot h[\kappa(\zeta+\theta)]-\mathbf{E}^{-1}\\&\cdot\left[\kappa^{\delta}\mathbf{E}\left\{\sum_{l=0}^{\infty}\mathscr{A}_l+\frac{1}{2}\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}+\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}\right\}\right],\\\sum_{l=0}^{\infty}\mathbb{K}_{l+1}(\zeta,\rho)&=-\kappa^2\operatorname{cosec}h^2[\kappa(\zeta+\theta)]-\mathbf{E}^{-1}\\&\cdot\left[\kappa^{\delta}\mathbf{E}\left\{\sum_{l=0}^{\infty}\mathscr{B}_l+\sum_{l=0}^{\infty}\mathscr{C}_l-\frac{1}{2}\frac{\partial^2\mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right\}\right].\end{aligned}\tag{52}$$

By the comparison of both sides of equation (52), we obtain

$$\begin{aligned}\mathbb{J}_0(\zeta,\rho)&=\lambda-\kappa\cot h[\kappa(\zeta+\theta)],\\\mathbb{K}_0(\zeta,\rho)&=-\kappa^2\operatorname{cosec}h^2[\kappa(\zeta+\theta)],\\\mathbb{J}_1(\zeta,\rho)&=-\lambda\kappa^2\operatorname{cosec}h^2[\kappa(\zeta+\theta)]\frac{\rho^{\delta}}{\Gamma(\delta+1)},\\\mathbb{K}_1(\zeta,\rho)&=-\lambda\kappa^2\operatorname{cosec}h^2[\kappa(\zeta+\theta)]\cot h[\kappa(\zeta+\theta)]\frac{\rho^{\delta}}{\Gamma(\delta+1)},\\\mathbb{J}_2(\zeta,\rho)&=\lambda\kappa^4\operatorname{cosec}h^2[\kappa(\zeta+\theta)]\\&\quad\cdot\left\{2\lambda\kappa\frac{\rho^{3\delta}}{\Gamma(3\delta+1)}-\left(3\cot h^2([\kappa(\zeta+\theta)]-1)\right)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}\right\},\\\mathbb{K}_2(\zeta,\rho)&=\left[2\lambda\kappa^5\operatorname{cosec}h^2[\kappa(\zeta+\theta)]\right]\\&\quad\cdot\left[\lambda\kappa\operatorname{cosec}h^2\left(3\cot h^2([\kappa(\zeta+\theta)]-1)\right)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}\right.\\&\quad+\frac{2\lambda\kappa\operatorname{cosec}h^2\cot h^2([\kappa(\zeta+\theta)])\rho^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)}\\&\quad\left.-2\lambda\coth\left(3\operatorname{cosec}h^2([\kappa(\zeta+\theta)]-1)\right)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}\right].\end{aligned}\tag{53}$$

Thus, for $\mathbb{J}_l$ and $\mathbb{K}_l$ with $(l\geq 3)$, the remaining components are easily computable. So, the solution in series form is as follows:

$$\begin{aligned}\mathbb{J}(\zeta,\rho)&=\sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho)=\mathbb{J}_0(\zeta,\rho)+\mathbb{J}_1(\zeta,\rho)+\mathbb{J}_2(\zeta,\rho)+\cdots,\\\mathbb{J}(\zeta,\rho)&=\lambda-\kappa\cot h[\kappa(\zeta+\theta)]-\lambda\kappa^2\operatorname{cosec}h^2[\kappa(\zeta+\theta)]\\&\quad\cdot\frac{\rho^{\delta}}{\Gamma(\delta+1)}+\lambda\kappa^4\operatorname{cosec}h^2[\kappa(\zeta+\theta)]\\&\quad\cdot\left\{2\lambda\kappa\left\{(1-\delta)^2 3\delta\rho+(1-\delta)^3+\frac{3\delta^2(1-\delta)\rho^2}{2}+\frac{\delta^3\rho^3}{3!}\right\}\right.\\&\quad\left.-\left(3\cot h^2([\kappa(\zeta+\theta)]-1)\right)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}\right\}+\cdots.\end{aligned}$$

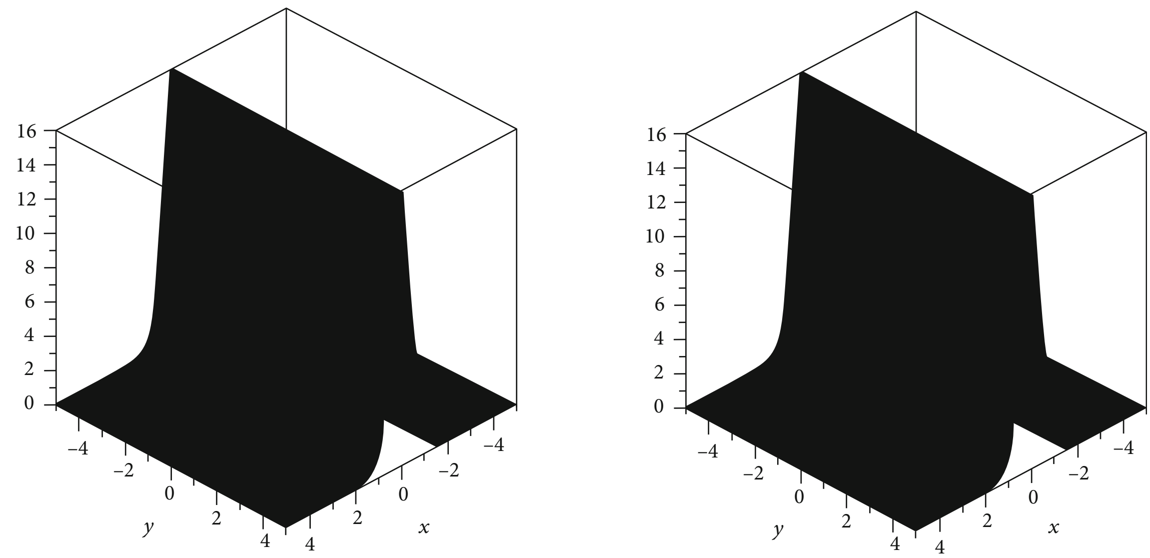

Figure 5: Graphical view of the exact and analytical solution for $\mathbb{K}(\zeta,\rho)$ at $\delta=1$ of system 1.

$$
\begin{aligned}
\mathbb{K}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{K}_l(\zeta,\rho)=\mathbb{K}_0(\zeta,\rho)+\mathbb{K}_1(\zeta,\rho)+\mathbb{K}_2(\zeta,\rho)+\cdots,\\
\mathbb{K}(\zeta,\rho) &= -\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta+\theta)]-\lambda\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta+\theta)] \cot h\\
&\quad\cdot[\kappa(\zeta+\theta)]\frac{\rho^{\delta}}{\Gamma(\delta+1)}+\left[2\lambda\kappa^5 \operatorname{cosec} h^2[\kappa(\zeta+\theta)]\right]\\
&\quad\cdot\left[\lambda\kappa \operatorname{cosec} h^2\left(3\cot h^2([\kappa(\zeta+\theta)]-1)\right)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}\right.\\
&\quad+\frac{2\lambda\kappa \operatorname{cosec} h^2 \cot h^2([\kappa(\zeta+\theta)])\rho^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)}\\
&\quad-2\lambda\cot h\left(3\operatorname{cosec} h^2([\kappa(\zeta+\theta)]-1)\right)\\
&\quad\left.\cdot\left((1-\delta)^2+2\delta(1-\delta)\rho+\frac{\delta^2\rho^2}{2}\right)\right]+\cdots.
\end{aligned}
\tag{54}
$$

4.4. Solution by Means of EDM$_{ABC}$. The solutions in series form for the unknown function $\mathbb{J}(\zeta,\rho)$ and $\mathbb{K}(\zeta,\rho)$ are stated as follows:

$$
\begin{aligned}
\mathbb{J}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho),\\
\mathbb{K}(\zeta,\rho) &= \sum_{l=0}^{\infty}\mathbb{J}_l(\zeta,\rho).
\end{aligned}
\tag{55}
$$

The nonlinear terms by means of Adomian polynomials are stated as $\mathbb{J}\mathbb{J}_\zeta=\sum_{l=0}^{\infty}\mathscr{A}_l$ and $\mathbb{J}^2\mathbb{J}_\zeta=\sum_{l=0}^{\infty}\mathscr{B}_l$; thus, by means of these terms, equation (50) can be determined as follows:

$$
\begin{aligned}
\sum_{l=0}^{\infty}\mathbb{J}_{l+1}(\zeta,\rho) &= \lambda-\kappa\cot h[\kappa(\zeta+\theta)]+\mathbf{E}^{-1}\left[\frac{\ell^{\delta}\left(\kappa^{\delta}+\delta\left(\ell^{\delta}-\kappa^{\delta}\right)\right)}{\kappa^{2\delta}}\mathbf{E}\left\{\sum_{l=0}^{\infty}\mathscr{A}_l+\frac{1}{2}\frac{\partial\mathbb{J}(\zeta,\rho)}{\partial\zeta}+\frac{\partial\mathbb{K}(\zeta,\rho)}{\partial\zeta}\right\}\right],\\
\sum_{l=0}^{\infty}\mathbb{K}_{l+1}(\zeta,\rho) &= -\kappa^2\operatorname{cosec} h^2[\kappa(\zeta+\theta)]+\mathbf{E}^{-1}\left[\frac{\ell^{\delta}\left(\kappa^{\delta}+\delta\left(\ell^{\delta}-\kappa^{\delta}\right)\right)}{\kappa^{2\delta}}\mathbf{E}\left\{\sum_{l=0}^{\infty}\mathscr{B}_l+\sum_{l=0}^{\infty}\mathscr{C}_l-\frac{1}{2}\frac{\partial^2\mathbb{K}(\zeta,\rho)}{\partial\zeta^2}\right\}\right].
\end{aligned}
\tag{56}
$$

By the comparison of both sides of equation (56), we obtain

$$
\begin{aligned}
\mathbb{J}_0(\zeta,\rho) &= \lambda-\kappa\cot h[\kappa(\zeta+\theta)],\\
\mathbb{K}_0(\zeta,\rho) &= -\kappa^2\operatorname{cosec} h^2[\kappa(\zeta+\theta)],\\
\mathbb{J}_1(\zeta,\rho) &= -\lambda\kappa^2\operatorname{cosec} h^2[\kappa(\zeta+\theta)]\left(1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{K}_1(\zeta,\rho) &= -\lambda\kappa^2\operatorname{cosec} h^2[\kappa(\zeta+\theta)]\cot h[\kappa(\zeta+\theta)]\\
&\quad\cdot\left(1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right),\\
\mathbb{J}_2(\zeta,\rho) &= \lambda\kappa^4\operatorname{cosec} h^2[\kappa(\zeta+\theta)]\\
&\quad\cdot\left\{2\lambda\kappa\frac{\rho^{3\delta}}{\Gamma(3\delta+1)}-\left(3\cot h^2([\kappa(\zeta+\theta)]-1)\right)\right.\\
&\quad\left.\cdot\left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}+2\delta(1-\delta)\frac{\rho^{\delta}}{\Gamma(\delta+1)}+(1-\delta)^2\right]\right\},
\end{aligned}
$$

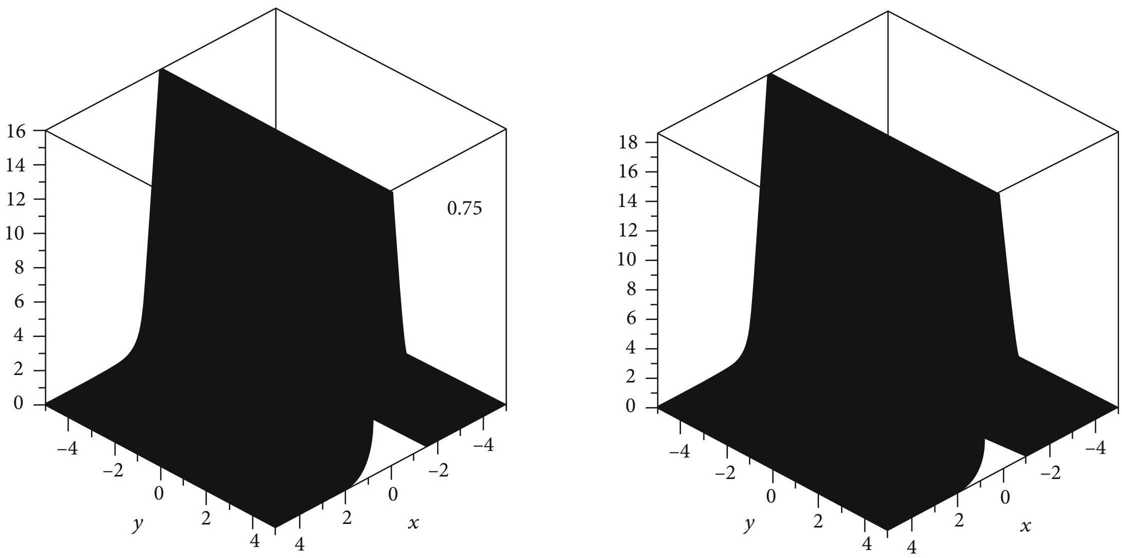

Figure 6: Graphical view of the analytical solution for $\mathbb{K}(\zeta, \rho)$ at $\delta = 0.8,0.6$ of system 1.

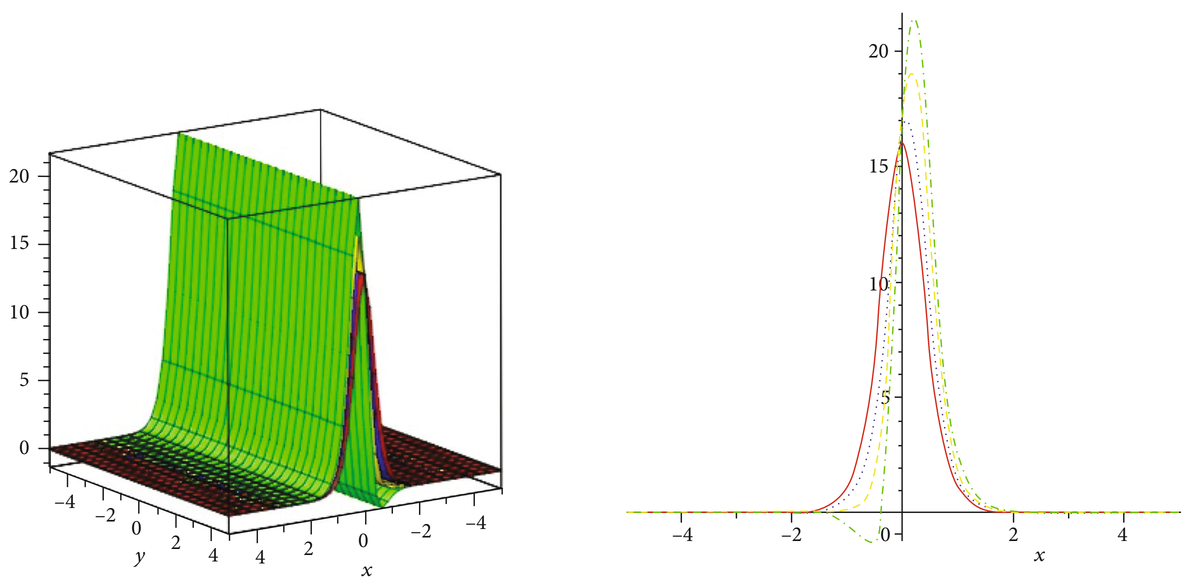

Figure 7: Graphical view of analytical solution at various values of δ for $\mathbb{K}(\zeta, \rho)$ of system 1.

Table 1: Proposed method solution for $\mathbb{J}(\zeta, \rho)$ at different fractional orders of problem 1.

(ζ, ρ)	$\mathbb{J}(\zeta, \rho)$ at $\delta = 0.5$	$\mathbb{J}(\zeta, \rho)$ at $\delta = 0.75$	$(ETDM_{ABC})$ at $\delta = 1$	$(ETDM_{C})$ at $\delta = 1$	Exact result
(0.2,0.01)	2.427701	2.427745	2.427806	2.427806	2.427806
(0.4,0.02)	4.701611	4.711644	4.711736	4.711736	4.711736
(0.6,0.03)	6.057710	6.057766	6.157882	6.157882	6.157882
(0.2,0.01)	2.414601	2.415654	2.414781	2.414781	2.414781
(0.4,0.02)	4.602065	4.702334	4.702226	4.702226	4.702226
(0.6,0.03)	6.053106	6.153617	6.153237	6.153237	6.153237
(0.2,0.01)	2.426313	2.426388	2.426426	2.426426	2.426426
(0.4,0.02)	4.700712	4.700746	4.710841	4.710841	4.710841
(0.6,0.03)	6.057423	6.057369	6.157403	6.157403	6.157403
(0.2,0.01)	2.324601	2.425646	2.425742	2.425742	2.425742
(0.4,0.02)	4.710311	4.710355	4.710403	4.710403	4.710403
(0.6,0.03)	6.058112	6.157050	6.157150	6.157150	6.157150
(0.2,0.01)	2.426076	2.425223	2.425158	3.536168	3.536168
(0.4,0.02)	4.700002	4.710010	4.810046	4.810046	4.810046
(0.6,0.03)	6.058000	6.057002	6.057015	6.057015	6.057015

Table 2: Proposed method solution for $\mathbb{K}(\zeta, \rho)$ at different fractional orders of problem 1.

(ζ, ρ)	$\mathbb{K}(\zeta, \rho)$ at δ =0.5	$\mathbb{K}(\zeta, \rho)$ at δ =0.75	$(ETDM_{ABC})$ at $\delta = 1$	$(ETDM_{C})$ at $\delta = 1$	Exact result
(0.2,0.01)	12.58221	12.571167	12.581150	12.581150	12.581150
(0.4,0.02)	7.835001	7.835012	7.835060	7.835060	7.835060
(0.6,0.03)	3.771048	3.772101	3.771122	3.771122	3.771122
(0.2,0.01)	12.61072	12.630787	12.610884	12.610884	12.610884
(0.4,0.02)	7.857405	8.857468	7.857542	7.857542	7.857542
(0.6,0.03)	3.785415	3.785477	3.785514	3.785514	3.785514
(0.2,0.01)	12.58215	12.582201	12.585430	12.585430	12.585430
(0.4,0.02)	8.837267	7.837311	7.838556	7.838556	7.838556
(0.6,0.03)	3.771568	3.771604	3.772651	3.772651	3.772651
(0.2,0.01)	12.58320	12.58321	12.583270	12.583270	12.583270
(0.4,0.02)	7.838476	8.838511	7.838463	7.838463	7.838463
(0.6,0.03)	3.772401	4.772434	4.772464	4.772464	4.772464
(0.2,0.01)	12.58427	12.584301	12.584310	12.584310	12.584310
(0.4,0.02)	7.840625	7.840667	8.840713	8.840713	8.840713
(0.6,0.03)	3.773177	3.773215	4.773278	4.773278	4.773278

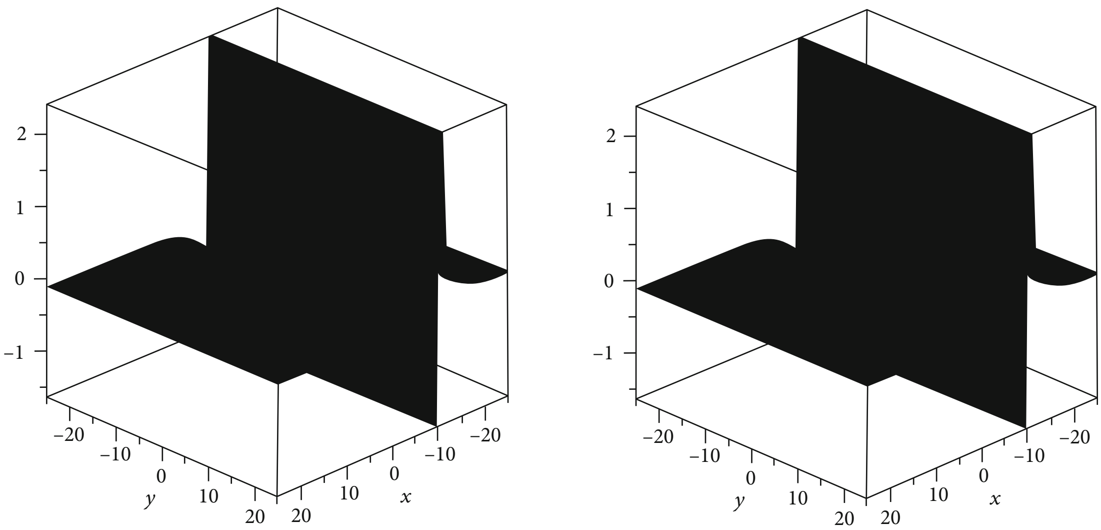

Figure 8: Graphical view of the exact and analytical solution for $\mathbb{J}(\zeta,\rho)$ at $\delta = 1$ of system 2.

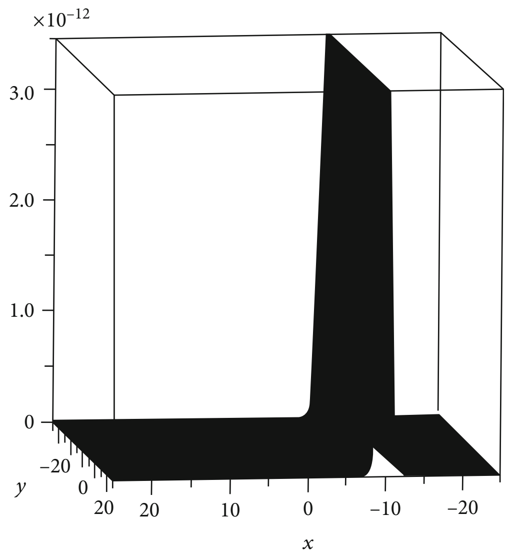

Figure 9: Graphical view of the absolute error for $\mathbb{J}(\zeta,\rho)$ of system 2.

$$\begin{aligned}\mathbb{K}_2(\zeta,\rho) = {} & \left[2\lambda\kappa^5 \operatorname{cosec} h^2[\kappa(\zeta+\theta)]\right] \\ & \cdot \left[\lambda\kappa \operatorname{cosec} h^2\left(3 \cot h^2([\kappa(\zeta+\theta)]-1)\right)\frac{\rho^{2\delta}}{\Gamma(2\delta+1)}\right. \\ & + \frac{2\lambda\kappa \operatorname{cosec} h^2 \cot h^2([\kappa(\zeta+\theta)])\rho^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)} \\ & - 2\lambda \cot h\left(3 \operatorname{cosec} h^2([\kappa(\zeta+\theta)]-1)\right) \\ & \left.\cdot\left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^{\delta}}{\Gamma(\delta+1)} + (1-\delta)^2\right]\right].\end{aligned} \tag{57}$$

Thus, for $\mathbb{J}_l$ and $\mathbb{K}_l$ with $(l \geq 3)$, the remaining components are easily computable. So, the solution in series form is as follows:

$$\mathbb{J}(\zeta,\rho) = \sum_{l=0}^{\infty} \mathbb{J}_l(\zeta,\rho) = \mathbb{J}_0(\zeta,\rho) + \mathbb{J}_1(\zeta,\rho) + \mathbb{J}_2(\zeta,\rho) + \cdots,$$

$$\begin{aligned}\mathbb{J}(\zeta,\rho) = {} & \lambda - \kappa \cot h[\kappa(\zeta+\theta)] - \lambda\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta+\theta)] \\ & \cdot \left(1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right) + \lambda\kappa^4 \operatorname{cosec} h^2[\kappa(\zeta+\theta)] \\ & \cdot \left\{2\lambda\kappa\left\{(1-\delta)^2 3\delta\rho + (1-\delta)^3 + \frac{3\delta^2(1-\delta)\rho^2}{2} + \frac{\delta^3\rho^3}{3!}\right\}\right. \\ & - \left(3 \cot h^2([\kappa(\zeta+\theta)]-1)\right) \\ & \left.\cdot\left[\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^{\delta}}{\Gamma(\delta+1)} + (1-\delta)^2\right]\right\} + \cdots.\end{aligned}$$

$$\mathbb{K}(\zeta,\rho) = \sum_{l=0}^{\infty} \mathbb{K}_l(\zeta,\rho) = \mathbb{K}_0(\zeta,\rho) + \mathbb{K}_1(\zeta,\rho) + \mathbb{K}_2(\zeta,\rho) + \cdots,$$

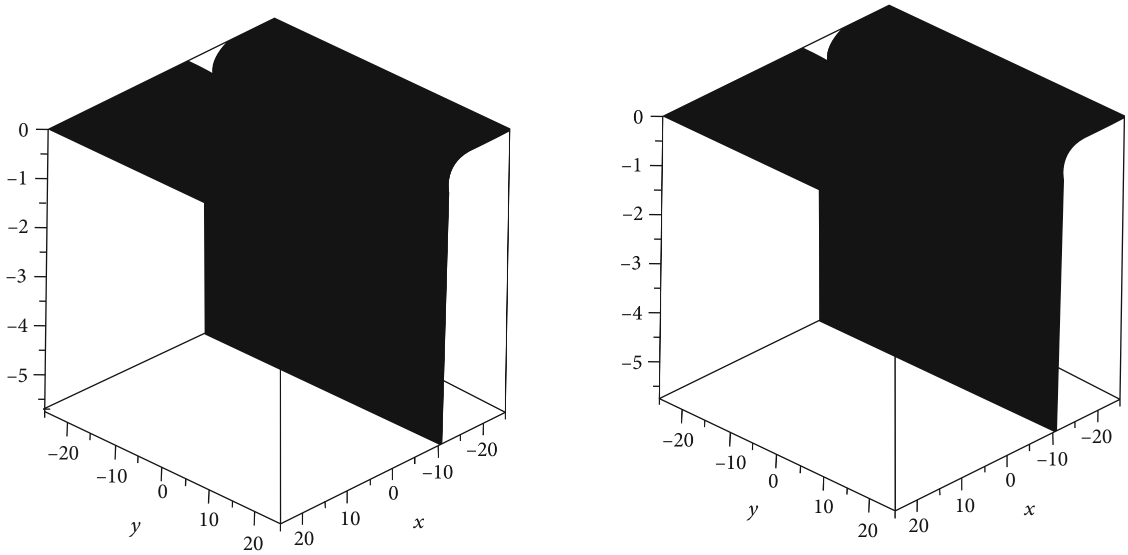

Figure 10: Graphical view of the exact and analytical solution for $\mathbb{K}(\zeta, \rho)$ at $\delta = 1$ of system 2.

$$
\begin{aligned}
\mathbb{K}(\zeta, \rho) = & -\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta+\theta)] - \lambda\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta+\theta)] \cot h[\kappa(\zeta+\theta)] \\
& \cdot \left(1 - \delta + \frac{\delta \rho^{\delta}}{\Gamma(\delta+1)}\right) + \left[2\lambda\kappa^5 \operatorname{cosec} h^2[\kappa(\zeta+\theta)]\right] \\
& \cdot \left[\lambda\kappa \operatorname{cosec} h^2\left(3 \cot h^2([\kappa(\zeta+\theta)] - 1)\right) \frac{\rho^{3\delta}}{\Gamma(3\delta+1)}\right. \\
& + \frac{2\lambda\kappa \operatorname{cosec} h^2 \cot h^2([\kappa(\zeta+\theta)])\rho^{3\delta}}{\Gamma(\delta+1)\Gamma(3\delta+1)} \\
& - 2\lambda \cot h\left(3 \operatorname{cosec} h^2([\kappa(\zeta+\theta)] - 1)\right) \\
& \left.\cdot \left[\frac{\delta^2 \rho^{2\delta}}{\Gamma(2\delta+1)} + 2\delta(1-\delta)\frac{\rho^{\delta}}{\Gamma(\delta+1)} + (1-\delta)^2\right]\right] + \cdots.
\end{aligned} \tag{58}
$$

We obtain the below series form solution at integer order $\delta = 1$, $\kappa = 0.1$, $\lambda = 0.005$, $\theta = 10$, as follows:

$$
\begin{aligned}
\mathbb{J}(\zeta, \rho) = & \ 0.005 - 0.1 \cot h(0.1\zeta + 10) \\
& - 0.0005 \operatorname{cosec} h^2(0.1\zeta + 10)\rho + 5 \\
& \times 10^{-7} \operatorname{cosec} h^2(0.1\zeta + 10) 0.003\rho^3 \\
& - 0.5\left(3 \cot h^2(0.1\zeta + 10) - 1.\right)\right)\rho^2, \\
\mathbb{K}(\zeta, \rho) = & -0.01 \operatorname{cosec} h^2(0.1\zeta + 10) \\
& - 0.000010 \operatorname{cosec} h^2(0.1\zeta + 10) \\
& \times \cot h(0.1\zeta + 10)\rho + 1.0 \\
& \times 10^{-7} \operatorname{cosec} h^2(0.1\zeta + 10) \\
& \times \left[8.3 \times 10^{-5}\rho^3 \operatorname{cosec} h^2(0.1\zeta + 10)\right. \\
& \cdot (3 \cot h(0.1\zeta + 10) - 1) - \rho^2 \cot h(0.1\zeta + 10) \\
& \cdot \left(3 \operatorname{cosec} h^2(0.1\zeta + 10) - 1\right) + 1.6 \\
& \left.\times 10^{-4}\rho^3 \operatorname{cosec} h^2(0.1\zeta + 10) \cot h(0.1\zeta + 10)\right].
\end{aligned} \tag{59}
$$

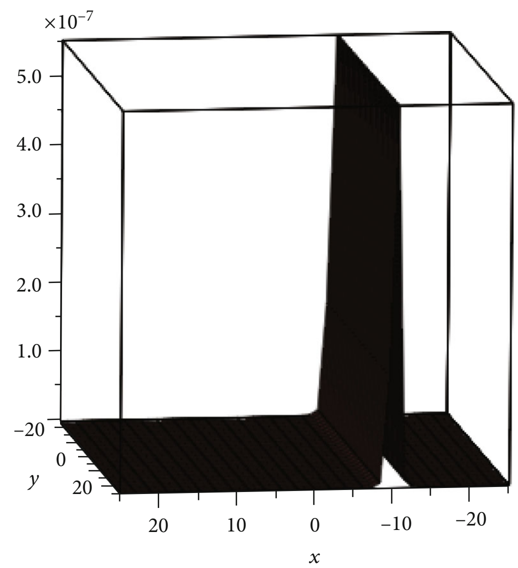

Figure 11: Graphical view of the absolute error for $\mathbb{K}(\zeta, \rho)$ of system 2.

The exact solution of equation (45) at $\delta = 1$ and taking $\kappa = 0.1$, $\lambda = 0.005$, $\theta = 10$,

$$
\begin{aligned}
\mathbb{J}(\zeta, \rho) &== \lambda - \kappa \cot h[\kappa(\zeta + \theta - \lambda\rho)], \\
\mathbb{K}(\zeta, \rho) &= -\kappa^2 \operatorname{cosec} h^2[\kappa(\zeta + \theta - \lambda\rho)].
\end{aligned} \tag{60}
$$

Figure 8 shows the graphical view of the exact and analytical solution for $\mathbb{J}(\zeta, \rho)$ at $\delta = 1$ of system 2, and Figure 9 shows the absolute error for $\mathbb{J}(\zeta, \rho)$ of system 2. Similarly, Figure 10 represents the exact and analytical solution for $\mathbb{K}(\zeta, \rho)$ at $\delta = 1$ of system 2 and Figure 11 of the

Table 3: Proposed method solution for $\mathbb{J}(\zeta, \rho)$ at different fractional orders of problem 2.

(ζ, ρ)	$\mathbb{J}(\zeta, \rho)$ at $\delta = 0.5$	$\mathbb{J}(\zeta, \rho)$ at $\delta = 0.75$	$(ETDM_{ABC})$ at $\delta = 1$	$(ETDM_C)$ at $\delta = 1$	Exact result
(0.2,0.01)	−0.134802	−0.132796	−0.134882	−0.134882	−0.134882
(0.4,0.01)	−0.133485	−0.132568	−0.133543	−0.123553	−0.123553
(0.6,0.01)	−0.132287	−0.133292	−0.132380	−0.132380	−0.132380
(0.2,0.02)	−0.133907	−0.142898	−0.134992	−0.134992	−0.134992
(0.4,0.02)	−0.132497	−0.132578	−0.133453	−0.133453	−0.133453
(0.6,0.02)	−0.123499	−0.133293	−0.132380	−0.132380	−0.132380
(0.2,0.03)	−0.124909	−0.145899	−0.134792	−0.134792	−0.134792
(0.4,0.03)	−0.124586	−0.132568	−0.133453	−0.133453	−0.133453
(0.6,0.03)	−0.111299	−0.133289	−0.132380	−0.132380	−0.132380
(0.2,0.04)	−0.134808	−0.142896	−0.134792	−0.134792	−0.134792
(0.4,0.04)	−0.132489	−0.132576	−0.133453	−0.133453	−0.133453
(0.6,0.04)	−0.144496	−0.135288	−0.132380	−0.132380	−0.132380
(0.2,0.05)	−0.145905	−0.135897	−0.134792	−0.134792	−0.134792
(0.4,0.05)	−0.187578	−0.132564	−0.133453	−0.133453	−0.133453
(0.6,0.05)	−0.133298	−0.133290	−0.132480	−0.132480	−0.132480

Table 4: Proposed method solution for $\mathbb{K}(\zeta, \rho)$ at different fractional orders of problem 2.

(ζ, ρ)	$\mathbb{K}(\zeta, \rho)$ at $\delta = 0.5$	$\mathbb{K}(\zeta, \rho)$ at $\delta = 0.75$	$(ETDM_{ABC})$ at $\delta = 1$	$(ETDM_C)$ at $\delta = 1$	Exact result
(0.2,0.01)	−0.005794	−0.005784	−0.006771	−0.006771	−0.006872
(0.4,0.01)	−0.005446	−0.005436	−0.006424	−0.006424	−0.006525
(0.6,0.01)	−0.005319	−0.005313	−0.006100	−0.006100	−0.006200
(0.2,0.02)	−0.005794	−0.005684	−0.006773	−0.006773	−0.006773
(0.4,0.02)	−0.005442	−0.005433	−0.006424	−0.006424	−0.006424
(0.6,0.02)	−0.005319	−0.005314	−0.006301	−0.006301	−0.006301
(0.2,0.03)	−0.005789	−0.005783	−0.006871	−0.006871	−0.006871
(0.4,0.03)	−0.005443	−0.005437	−0.006424	−0.006424	−0.006424
(0.6,0.03)	−0.005316	−0.005306	−0.006100	−0.006100	−0.006100
(0.2,0.04)	−0.005789	−0.005782	−0.006773	−0.006773	−0.006773
(0.4,0.04)	−0.005441	−0.005431	−0.006421	−0.006421	−0.006421
(0.6,0.04)	−0.005118	−0.005306	−0.006301	−0.006301	−0.006301
(0.2,0.05)	−0.005789	−0.005782	−0.006572	−0.006572	−0.006572
(0.4,0.05)	−0.005446	−0.005437	−0.006424	−0.006424	−0.006424
(0.6,0.05)	−0.005323	−0.005112	−0.006201	−0.006201	−0.006201

absolute error for $\mathbb{K}(\zeta, \rho)$ of system 2. Tables 3 and 4 show that the different fractional order of δ of system 2.

5. Conclusion

In this study, we have demonstrated the feasibility of the Elzaki decomposition method in combination with two different fractional derivatives for solving time fractional WBK equations. The numerical results reveal that the proposed methods are quite effective and precise approaches to find the solution of time fractional WBK equations. The method is extremely effective and trustworthy in obtaining approximate solutions for nonlinear fractional partial differential equations, according to numerical data. The proposed technique is an efficient and easy tool for investigating numerical solution of nonlinear coupled systems of fractional partial differential equations when compared to previous analytical techniques. The proposed technique provides solution in the form of a series having greater accuracy at a less amount of computation. Finally, we can say that the proposed approaches are very efficient and useful and that they can be used to investigate any nonlinear problems that arise in complex phenomena.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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