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A New Life-Time Model with Bathtub and Inverted Bathtub Failure Rate Function

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, a three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution is proposed. The extension was done using Zubair G-Family [1] of continuous probability distribution to extend the well knownKumaraswamy distribution to make it more flexible in modeling and predicting real world phenomenon. Some basic structural properties of the new distributions like cdf, pdf, quantile functions, moments, moment generating functions, characteristics functions and order statistics was obtained. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. Behavior of the hazard rate plot exhibit increase, decrease, Bathtub and inverted Bathtub shape. Maximum likelihood estimate was used to estimate the Z-Kum distribution parameters. Monte Carlo simulation was carried out to evaluate the performance of MLE method adopted. Result of the simulation studies indicates that MLE is good for the estimation of our distribution parameters. To compare the proposed model with the other fitted existing models, analytical measure of goodness of fit of some information criterion was considered using three real life data sets. From the results obtained, it is evident that our proposed model gives better fit than the other competing models and therefore, our proposed model provide greater flexibility in modeling real world phenomenon.

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1 Introduction

The quality of the procedures used in statistical analysis depends heavily on the assumed probability model or distribution that the random variable follows. Many lifetime data used for statistical analysis follow a particular probability distribution and therefore knowledge of the appropriate distribution that any phenomenon follows greatly improves the sensitivity, reliability and efficiency of the statistical analysis associated with it. Furthermore, it is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. Due to change in the world population and rapid global development in science and technology, developing new distributions which could better describe some of these phenomena and provide greater flexibility and wider acceptability in the modeling of lifetime data become an inevitable.

Recently, attempts have been made to define new models that extend well known distributions and provide a greater flexibility in modeling real data and to improve the goodness-of-fit of the generated family. For instance, Eugene et al. [2] introduced a new class of distributions generated from the beta distribution. Zografos and Balakrishnan (2009) proposed the gamma generated family. Bourguignon et al. [3] presented the Weibull-generated (*W-G*) family of distributions with two extra parameters, Kummer beta generalized family of distributions by Pescim et al. (2012), geometric exponential-Poisson family of distributions by Nadarajah et al. [4], exponentiated T-X family of distributions by Alzaghal et al.(2013), weibull generalized family of distributins by Bourguignionet al.[3];modified beta generalized family distributions by Nadarajah et al. (2014), kumaraswamyWeibul by Cordeiro et al. [5]kumaraswamygumbel by Cordeiro et al.[5], Kumaraswamy Birnbaum-Saunders by Saulo and Bourguibnon (2012), Kumaraswamy Pareto by Bourguibnon et al.[6], Kumaraswamy generalized Rayleigh by Gomes et al. (2014). Kumaraswamy inverse by Rayleigh Roges et al. (2014), Kumaraswamy modified inverse Weibull by Aryal and Elbatal[7], Kumaraswamy Laplace by Aryal[7], Kumaraswamy exponential-Weibull by Cordeiro et al. [5], Kumaraswamyexponentiated inverse Rayleigh by Haq[8]; Olalekan et al.[9]; Ayed et al.[10].

2 Some Existing Probability Distributions

2.1 Kumarasawamydistribution

The pdf and cdf of Kumarasawamy distribution are as given in (1) and (2) respectively.

$$G(x;a,b) = 1-\left(1-x^{a}\right)^{b}, \qquad 0<x<1,\ a>0, b>0. \quad , \tag{1}$$

$$g(x;a,b) = abx^{a-1}\left(1-x^{a}\right)^{b-1}, \quad 0<x<1,\ a>0, b>0. \tag{2}$$

Where a and b are two shape parameters.

2.2 Zubair G Family of distribution

A family of life distributions, called the Zubair-G family was introduced by Zubair (2018). The benefit of using this family is that its cdf has a closed form solution and capable of data modeling with monotonic and non-monotonic failure rates. The CDF and PDF of the new family defined by Zubair (2018) for random Variable X is given in (3) and (4) respectively.

$$F(x,\alpha\varphi)=\frac{exp\{\alpha G(x;\varphi)^2\}-1}{\exp(\alpha)-1}, \qquad \alpha>0,\ x\varepsilon\Re. \quad (3)$$

$$f(x,\alpha,\varphi)=\frac{2\alpha g(x;\phi)G(x;\varphi)exp\{\alpha G(x;\varphi)^2\}}{\exp(\alpha)-1}, \qquad \alpha>0, x\varepsilon\Re, \quad (4)$$

Where φ is vector of the baseline distribution parameter, α is the parameter of Zubair G-family $g(x;\phi)$ and $G(x;\varphi)$ are pdf and cdfof the baseline distribution respectively.

3 Proposed Distribution

3.1 Zubair-Kumarasawamy (Z-Kum) distribution

In this paper, Zubair-Kumarasawamy distribution (Z-Kum) was proposed as version of Kumarasawamy distribution (Kum distribution) is obtained by putting (1) into (3). Then

$$F(x;\alpha,a,b)_{Z-Kum}=\frac{\exp\left(\alpha(1-(1-x^a)^b)^2\right)-1}{\exp(\alpha)-1}\ 0<x<1,\ a>0,b>0. (4)$$

The corresponding probability density function (pdf) of Zubair–Kumarasawamy (Z-Kum) distribution denoted by $f(x;\alpha,a,b)_{Z-Kw}$ is obtained by differentiating (4) with respect to x.

$$f(x;\alpha,a,b)=\frac{d}{dx}\left[\frac{\exp\{\alpha(1-(1-x^a)^b)^2-1\}}{\exp(\alpha)-1}\right]$$
$$=\frac{1}{\exp(\alpha)-1}\frac{d}{dx}[\exp\{\alpha(1-(1-x^a)^b)^2-1\}](5)$$

$$putU=(1-(1-x^a)^b)^2 and \frac{du}{dx}=abx^{a-1}(1-x^a)^{b-1} ineq(5); weobtained$$

$$f(x;\alpha,a,b)_{Z-Kum}=\frac{2\alpha abx^{\alpha-1}\{(1-x^a)^{b-1}(1-(1-x^a)^b)\}\exp\{\alpha(1-(1-x^a)^b)^2\}}{\exp(\alpha)-1}(6)$$

$$0<x<1, a>0, b>0.$$

Theorem 3.1: To verify the proposed $f(x;\alpha,a,b)_{Z-Kum}$ is true probability density function. Then

$$\int_{x=0}^{1} f(x;\alpha,a,b)_{Z-Kum}dx=1 \quad (7)$$

Proof:$LetT_2=\int_{x=0}^{1} f(x;\alpha,a,b)_{Z-Kum}dx$

$$T_2=\int_{x=0}^{1}\frac{2\alpha abx^{\alpha-1}\{(1-x^a)^{b-1}(1-(1-x^a)^b)\}\exp\{\alpha(1-(1-x^a)^b)^2\}}{\exp(\alpha)-1}dx \quad (8)$$

$$Putu_2=1-x^a,\frac{du_2}{dx}=-ax^{a-1}\Rightarrow dx=-\frac{du}{ax^{a-1}} and\ 0<u_2<1\ in(8). weobtained$$

$$T_2 = \frac{2\alpha ab}{\exp(\alpha)-1}\int_{u=0}^{1} -\frac{(u_2^{b-1}-u_2^{b})\exp[\alpha(1-u_2^{b})^2]}{a}du \quad (9)$$

$$byputtingV = 1-u_2^{b}, \frac{dv}{du_2} = -bu_2^{b-1} \Rightarrow du_2 = -\frac{dv}{bu_2^{b-1}} and\ 0 < v < 1\ in(9), weget$$

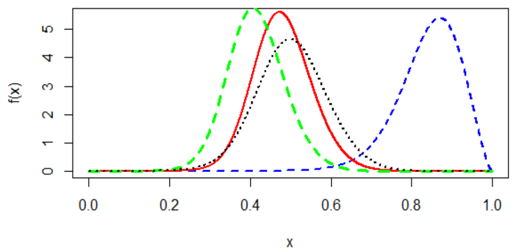
$$T_2 = \frac{2\alpha ab}{\exp(\alpha)-1}.\frac{1}{a}\left(\int_{v=0}^{1} -\frac{\exp(\alpha v^2)\,(v)}{b}dv\right)(10)$$

$$Letw = \alpha v^2, \frac{dw}{dv} = 2\alpha v \Rightarrow dv = \frac{dw}{2ab} and\ 0 < w < \alpha in(10), weget$$

$$T_2 = \frac{2\alpha ab}{\exp(\alpha)-1}.\frac{1}{ab}\left[\frac{\exp(w)}{2\alpha}\right]_0^{\alpha} = \frac{2\alpha ab}{\exp(\alpha)-1}.\frac{\exp(\alpha)-1}{2\alpha ab} = 1 \quad (11)$$

Hence, the proposed model of $(f(x;\alpha,a,b)_{Z-Kum})$ is true probability density function.

Figs. 1 and 2 below displayed the plots of the *pdf*and *cdf*of the Z-Bx distributionfor some selected parameter values respectively.



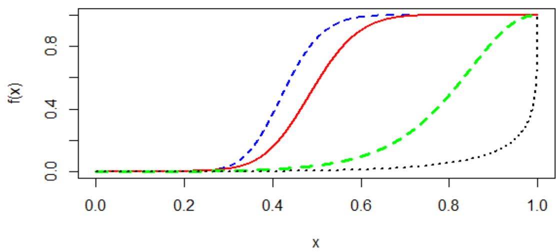


Fig. 1.Plot of Z-Kw PDFFig. 2. Plot of Z-Kw CDF

4 Properties of the proposed model

This section studies the statistical properties Z-Kum distribution such as the quantile function, order statistics, moments, moment generating function and characteristics function. Survival function, Hazard rate function, Reversed Hazard rate function, Cumulative hazard rate function are also discussed in details.

4.1 Quantile function of proposed distribution

The quantile function of Z-Kum distribution is obtained by inverting (4) as given in (12)

$$Q(U) = \left[1-\left\{1-\left(\frac{1}{\alpha}\ln(u(exp(\alpha)-1)+1)\right)^{\frac{1}{2}}\right\}^{\frac{1}{b}}\right]^{\frac{1}{a}} \quad (12)$$

$where, u \sim U(0,1)$

If we put $u\ =\ 0.25, 0.5\ and\ 0.75$ in (12) then first quartile, median and third quartile will be obtained.

4.2 Order statistics

The order statistics and their moments have great importance in many statistical problems and applications in reliability analysis and life testing. Suppose $X_1, X_2, \ldots\ldots\ldots\ldots, X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1n}, X_{2n}, \ldots\ldots\ldots\ldots, X_{in}$ denote the corresponding order statistic obtained from this sample. The i^{th} order statistic of the proposed distributions can be using (13)

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left(1 - F(x)\right)^{n-i} \quad (13)$$

Using bionomial expansion,

$$\left(1 - F(x)\right)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n-1}{k} F(x)^k \quad (14)$$

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \sum_{k=0}^{n-i} (-1)^k \binom{n-1}{k} F(x)^k \quad (15)$$

Inserting (3) and (6) in (15), the pdf of the ith order statistics can be given as in (16)

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} \left(\frac{2\alpha abx^{a-1} \left(\left(1 - x^a\right)^{b-1} - (1 - x^a)^{2b-1} \right) \exp\left(\alpha (1 - (1 - x^a)^b)^2 \right)}{\exp(\alpha) - 1} \right)$$
$$\times \left(\frac{\exp\left(\alpha (1 - (1 - x^a)^b)^2 \right) - 1}{\exp(\alpha) - 1} \right)^{i-1} \sum_{k=0}^{n-i} (-1)^k \binom{n-1}{k} \left(\frac{\exp\left(\alpha (1 - (1 - x^a)^b)^2 \right) - 1}{\exp(\alpha) - 1} \right)^k \quad (16)$$

4.3 Moments

The moments of a random variable are important in statistical inference. They are used to investigate important characteristics of a distribution such as the measures of central tendency, measures of dispersion and measures of shapes. In this subsection, the rth non-central moment of the Z-Kum random variable is derived.

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} X^r f(x, \phi)\, dx$$
$$= \int_{-\infty}^{\infty} X^r \frac{2\alpha abx^{\alpha-1} \{(1 - x^a)^{b-1} (1 - (1 - x^a)^b)\} \exp\{\alpha (1 - (1 - x^a)^b)^2\}}{\exp(\alpha) - 1} dx \qquad (17)$$

$$putting y = (1 - x)^b, \frac{dy}{dx} = -b(1 - x^a)^{b-1}. ax^{a-1} \Rightarrow dx = -\frac{dy}{b(1 - x^a)^{b-1} ax^{a-1}}\, in\ (17)$$

$$\mu_r' = -\frac{2\alpha}{\exp(\alpha) - 1} \int_0^1 \left[\left(y^{\frac{1}{b}} + 1 \right)^a \right]^r \exp[\alpha (1 - y)^2]\, dy \qquad (18)$$

$$put \left[\left(y^{\frac{1}{b}} + 1 \right)^a \right]^r = \sum_{j=0}^{r} \binom{r}{j} \left(y^{\frac{1}{b}} \right)^j (1)^{r-j}\ and\ \exp[\alpha(1 - y)^2] = \sum_{j=0}^{\infty} \frac{[\alpha (1 - y)^2]^j}{j!}\ in\ (18)$$

$$\mu_r' = -\frac{2\alpha}{\exp(\alpha) - 1} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \int_0^1 \left(y^{\left(\frac{1}{b}+1\right)-1} \right) (1 - y)^{(2j+1)-1} dy \qquad (19)$$

Hence,

$$\mu_r^{'} = -\frac{2\alpha a^j}{j!\exp(\alpha) - 1}\sum_{j=0}^{r}\sum_{j=0}^{\infty} B\left(\left(\frac{j}{b}+1\right), 2(j+1)\right) \quad (20)$$

$$where B\left(\left(\frac{j}{b}+1\right), 2(j+1)\right) = \int_0^1 \left(y^{\left(\frac{1}{b}+1\right)-1}\right)(1-y)^{(2j+1)-1}dy$$

where mean $=(x)=\mu_1^{/}, V(x)=\mu_2^{/}-\mu_1^{/^2}$, Skewness $= \dfrac{\mu_3^{/}-3\mu_2^{/}\mu_1^{/}+4\mu_1^{/^3}}{(\mu_2^{/}-\mu_1^{/^2})^{\frac{3}{2}}}$,

Kurtosis $= \dfrac{\mu_4^{/}-4\mu_3^{/}\mu_1^{/}+6\mu_1^{/^2}\mu_2^{/}-3\mu_1^{/^4}}{(\mu_3^{/}-\mu_1^{/})^2}$

4.4 Moment generating function of proposed distribution.

Theorem 4.1 Let X has a proposed distribution. Then the moment generating function is given by

$$M_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\left(\frac{t^r}{r!}\right)B\left(\left(\frac{j}{b}+1\right), 2\left(j+1\right)\right) \quad (21)$$

Proof:

$$M_X(t) = \mathrm{E}\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\ dx \quad (22)$$

Using Taylor series expansion, the moment generating function can be given as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}u_r \quad (23)$$

whereu_r is the rth non-central moment. Substituting the rth non-central moment as in (20) gives the moment generating function of Z-kumdistribution as

$$M_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\left(\frac{t^r}{r!}\right)B\left(\left(\frac{j}{b}+1\right), 2\left(j+1\right)\right) \quad (24)$$

Hence the proof.

4.5 Characteristic Function of proposed distribution

Theorem 4.2 Let X has a Z-kum distribution. Then the characteristic function is given by

$$\varphi_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\left(\frac{(it)^r}{r!}\right)B\left(\left(\frac{j}{b}+1\right), 2\left(j+1\right)\right) \quad (25)$$

Proof:

$$\phi_X(t) = \mathrm{E}\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)\, dx \quad (26)$$

Using Taylor series expansion, the characteristics function can be given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} u_r \quad (27)$$

Where u_r is the r^{th} non-central moment. Substituting the r^{th} non-central moment as in (20) gives the characteristics function of Z-kumdistribution as

$$\varphi_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \left(\frac{(it)^r}{r!} \right) B\left(\left(\frac{j}{b} + 1 \right), 2\left(j+1 \right) \right) \quad (28)$$

Hence the proof.

4.6 Survival Function of proposed distribution

Using (39) the survival function of Z-Kum distribution is obtain as in (30)

$$S(x) = 1 - F(x) \quad (29)$$

$$S(x) = \frac{\exp(\alpha) - \exp\left(\alpha + \alpha\left(-x^a + 1\right)^{2b} - 2\alpha\left(-x^a + 1\right)^b \right)}{\exp(\alpha) - 1} \quad (30)$$

4.7 Hazard Rate Function of Z-Kum distribution

The hazard rate functions of Z-Kum distribution is obtain using (31) as given in (32)

$$h(x) = \frac{f(x)}{S(x)} \quad (31)$$

$$h(x) = \frac{2\alpha abx^{a-1}\left(\left(1-x^a\right)^{b-1} (1-(1-x^a)^b) \right) \exp\left(\alpha(1-(1-x^a)^b)^2 \right)}{\exp(\alpha) - \exp\left(\alpha + \alpha\left(-x^a + 1\right)^{2b} - 2\alpha\left(-x^a + 1\right)^b \right)} \quad (33)$$

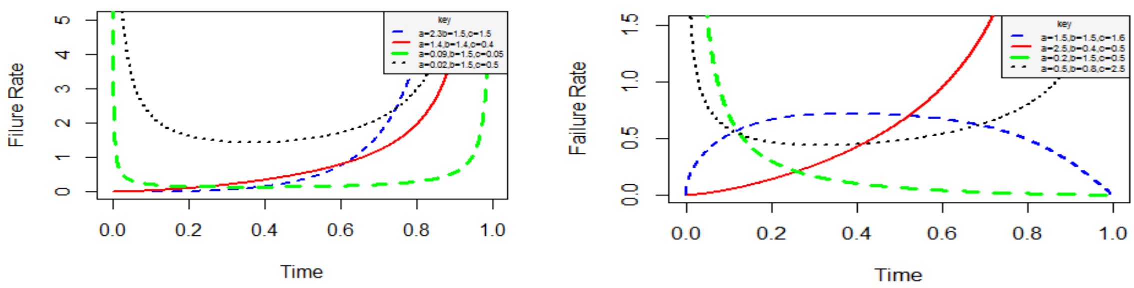


Fig. 3. Plot of of Z-kumhr

Plot of the hazard rate function of Z-Kum distribution for specific sets of parameter values is of Bath-Tub shape, inverted Bath-Tub shape, increasing and decreasing shape.

4.8 Reversed Hazard Rate Function of Z-Kum distribution

The Reversed hazard rate function of Z-Kum distribution is obtain using (33) as in (34)

$$r(x)=\frac{f(x)}{F(x)} \quad (33)$$

$$r(x)=\frac{2\alpha abx^{a-1}\left(\left(1-x^{a}\right)^{b-1}-(1-x^{a})^{2b-1}\right)\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)}{\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)-1} \quad (34)$$

4.9 Cumulative Hazard Rate Function of Z-Kum distribution

The cumulative hazard rate function of Z-Kum distribution is obtain using (35) as given in (36)

$$H(x)=-\log\left(1-F(x)\right) \quad (35)$$

$$H(x)=-\ln\left(\exp(\alpha)-\exp\left(\alpha+\alpha\left(-x^{a}+1\right)^{2b}-2\alpha\left(-x^{a}+1\right)^{b}\right)\right)-\ln\left(\exp(\alpha)-1\right) \quad (36)$$

5 Parameter Estimation and Simulation Studies of Z-Burr Distribution

5.1 Parameter estimation

To illustrate the applications of the developed distribution with regards to modeling real data sets, it is vital to develop estimators for estimating the parameters of the distribution. In this section, estimators are developed for estimating the parameters of the new distribution using the well known method of maximum likelihood estimate (MLE).

Let $X_1, X_2 ....X_n$ be a random sample from Z-Kum distribution with unknown parameter vector $\phi=(\alpha,a,b,)^{T}$, the likelihood function of the distribution is obtain using (37)

$$L(\phi)=\prod_{i=1}^{n} f(x_i,\phi)$$
$$=\prod_{i=1}^{n}\left[\frac{2\alpha abx^{\alpha-1}\{(1-x^{a})^{b-1}(1-(1-x^{a})^{b})\}\exp\{\alpha(1-(1-x^{a})^{b})^{2}\}}{\exp(\alpha)-1}\right] \quad (39)$$

$$L(\phi)$$
$$=\left[\frac{2\alpha ab}{\exp(\alpha)-1}\right]^{n}\left[\left(\prod_{i=1}^{n} x_i^{a-1}\right)\left(\prod_{i=1}^{n}(1-x^{a})^{b-1}\right)\prod_{i=1}^{n}(1-x^{a})^{2b-1}\exp\left\{\alpha^{n}\sum_{i=1}^{n}(1-(1-x^{a})^{b})^{2}\right\}\right]$$

Taking natural log likelihood is thus obtained as

$$\ell(\phi) = \log\{L(\phi)\}$$

$$= \begin{bmatrix} n(ln2 + ln\alpha + lnb) + (a-1)\sum_{i=1}^{n} lnx_i + (b-1)\left(\sum_{i=1}^{n} \ln(1-x^a)\right) \\ +(2b-1)\left(\sum_{i=1}^{n} \ln(1-x^a)\right) + \alpha^n \sum_{i=1}^{n}(1-(1-x_i^a)^b)^2 - nln(\exp(\alpha)-1) \end{bmatrix} \ldots \quad (40)$$

After differentiating (40) with respect to parameters $\alpha, a, and\ b$, we get

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + n\alpha^{n-1}\sum_{i=1}^{n}(1-(1-x_i^a)^b)^2 - \frac{nexp(\alpha)}{\exp(\alpha)-1}$$

$$= \frac{n}{\alpha}\left[1 + \alpha^n \sum_{i=1}^{n}(1-(1-x_i^a)^b)^2\right] - \frac{nexp(\alpha)}{\exp(\alpha)-1} = 0 \quad (41)$$

$$\frac{\partial \ell}{\partial a} = \begin{bmatrix} \frac{n}{a} + \sum_{i=1}^{n} lnx_i - (b-1)\sum_{i=1}^{n}\frac{x_i^a lna}{1-x_i^a} - (2b-1)\sum_{i=1}^{n}\frac{lna}{1-x_i^a} \\ +2\alpha^n b\sum_{i=1}^{n}[1-(1-x_i^a)^b](1-x_i^a)^{b-1}x_i^a lna \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n}{a} + \sum_{i=1}^{n} lnx_i - (b-1)\sum_{i=1}^{n}\frac{x_i^a lna}{1-x_i^a} - (2b-1)\sum_{i=1}^{n}\frac{lna}{1-x_i^a} \\ +2\alpha^n b\sum_{i=1}^{n}\left[\frac{(1-x_i^a)^b - (1-x_i^a)^{2b}}{1-x_i^a}\right]x_i^a lna \end{bmatrix} = 0 \quad (42)$$

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} ln(1-x_i^a) + 2\sum_{i=1}^{n} ln(1-x_i^a) - 2\alpha^n \sum_{i=1}^{n}(1-(1-x_i^a)^b)(1-x_i^a)^b \ln(1-x_i^a)$$

$$= \frac{n}{b} + 3\sum_{i=1}^{n} ln(1-x_i^a) - 2\alpha^n \sum_{i=1}^{n}[(1-x_i^a)^b - (1-x_i^a)^{2b}]\ln(1-x_i^a) = 0 \quad \ldots (43)$$

Equations (41), (42) and (43) cannot be solved analytically, statistical software like R can be used to simultaneously solve them numerically using iterative methods. Solutions of these equations provides the maximum likelihood estimate $\hat{\phi} = (\hat{\alpha}, \hat{a}, \hat{b})$ of $\phi = (\alpha, a, b)^T$

5.2 Simulation studies

The performance of the maximum likelihood estimates for the Z-Kum distribution parameters was evaluated using Monte Carlo simulation for a three parameter combinations. Different sample sizes (n = 50, 75 and 100) and some selected parameter values (α = 0.09, a = 0.08, b = 0.09) were used to perform the simulation. Result of the simulation is presented in the Table 1.

Table 1. Average MLEs, variance and MSE of the MLEs of parameters of Z-Kum distribution with actual parameter values (α = 0.09, a = 0.08, b = 0.09)

N	Estimates			Variance			MSE		
	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$
50	0.0724	0.0880	0.0980	0.0012	0.0003	0.0003	0.0015	0.5071	0.0003
75	0.0735	0.0874	0.0975	0.0011	0.0002	0.0002	0.0014	0.5069	0.0003
100	0.0767	0.0860	0.0964	0.0009	0.0002	0.0002	0.0011	0.5061	0.0002

5.3 Model comparison and selection criteria

To show how applicable and flexible our proposed model is, its performance is compared with other established models with reference to information lost. So, we tend to use information criteria techniques and goodness-of-fit statistics that correct model for complexity, to constrain the model from over fitting to assess the most effective model from a range of different models which can have different number of parameters.

In this case, we will consider the generally well known criteria such as Akaike Information Critareion (AIC), the Bayesian Information Criterion (BIC), the ConsistantAkaike Information Cretarion (CAIC) and Hannan-Quinn Information Criterion (HQIC) and illustrate the flexibility and applicability Z-Kum distribution, using three (3) real life data set.

Data set 1:

This data was used and analyze byMusa et al. [11]

0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454, 0.38392

Data set 2:

This data was used and analyze byMusa et al.[11]

0.42909, 0.83559, 0.85818, 0.79042, 0.67751, 0.99368, 0.88076, 0.88076, 0.93722, 0.74526, 0.76784, 0.82429, 0.77913, 0.68879, 0.98238, 0.71138, 0.76784, 0.51942, 0.77913, 0.70009, 0.54200, 0.75655, 0.86947, 0.99368, 0.76784, 0.92593, 0.80172, 0.46296, 0.76784, 0.76784, 0.48555, 0.89205, 0.36134, 0.65492, 0.79042, 0.84688, 0.80172, 0.64363, 0.42909, 0.74526, 0.80172, 0.48555, 0.67751, 0.75655, 0.47425, 0.94851, 0.92593, 0.63234, 0.93722, 0.73397, 0.71138, 0.90334, 0.72267, 0.99368, 0.63234, 0.45167, 0.65492, 0.92593, 0.41779, 0.72267, 0.75655, 0.47425, 0.94851, 0.48555, 0.63234, 0.54201, 0.89205, 0.80172, 0.65492, 0.46296, 0.75655, 0.84688, 0.47425, 0.65492, 0.51942, 0.39521, 0.91463, 0.37263, 0.66621, 0.49684, 0.86947, 0.82429, 0.63234, 0.41779, 0.74526, 0.80172, 0.12421, 0.16938, 0.15808, 0.09033, 0.88076, 0.37263, 0.66621, 0.18067, 0.85818, 0.83559, 0.64363, 0.49684, 0.76784, 0.77913, 0.89205, 0.35005, 0.99368, 0.60976, 0.75655, 0.77913, 0.65492, 0.39521, 0.74526, 0.82429, 0.92593, 0.97109, 0.68879, 0.94851, 0.7904, 0.99368, 0.71138, 0.49684, 0.06775, 0.91463, 0.97109, 0.91463, 0.86947, 0.76784, 0.86947, 0.79042, 0.79042, 0.41779, 0.77913, 0.99368, 0.51942, 0.67751, 0.84688, 0.80172, 0.90334, 0.80172, 0.90334, 0.71138, 0.63234, 0.74526, 0.54201, 0.39295, 0.76784, 0.71138, 0.67751, 0.63234, 0.77913, 0.85818, 0.63234, 0.99368, 0.55329, 0.75655, 0.82429, 0.37263, 0.56459, 0.15808, 0.45167, 0.64363, 0.67751, 0.99368, 0.92593, 0.67751, 0.84689, 0.68879, 0.76784, 0.50813, 0.68879, 0.82429, 0.67751, 0.28229, 0.49684, 0.62105, 0.66621, 0.62105, 0.86947, 0.89205, 0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454, 0.38392

Data set 3:

The following data was used and analyze by Saboor et al.[12]. It consist 48 rock samples from petroleum reservoir obtained from the measurements on petroleum rock samples

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470

Table 2. Performance of Z-Kum distribution's goodness of fit using data set 1

	MODEL		
	Z-KUM	**LIB-KUM**	**KUM**
AIC	-101.1618	-97.37334	-79.19838
CAIC	-99.82845	-96.04001	-77.86504
BIC	-97.88865	-94.10021	-75.92525
HQIC	-100.3907	-96.60229	-78.42733
Rank	**1**	**2**	**3**

Table 3. Performance of Z-Kum distribution's goodness of fit using data set 2

	MODEL		
	Z-KUM	**LIB-KKUM**	**KUM**
AIC	-977.89	-670.72	-816.81
CAIC	-977.78	-670.61	-816.70
BIC	-967.70	-660.52	-806.61
HQIC	-973.78	-666.61	-812.69
Rank	**1**	**2**	**3**

Table 4. Performance of Z-Kum distribution's goodness of fit using data set 3

	MODEL		
	Z-KUM	**LIB-KUM**	**KUM**
AIC	-134.4426	-126.1913	-118.3165
CAIC	-134.3220	-126.0807	-118.2060
BIC	-124.2481	-115.9968	-108.1221
HQIC	-130.3263	-122.0749	-114.2002
Rank	**1**	**2**	**3**

It can be seen from Tables 2, 3 and 4 that based on the values of the information criterion from the three different real life data sets, Z-Kum distribution having the less values performed better than the other two distributions in term of fitting/modeling real life data.

6 Discussion and Conclusion

In this paper, we developed new three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution. The extension was done using Zubair G-Family [1] of continuous probability distribution to extend well known Kumaraswamy distribution to make it more flexible in modeling and predicting real world phenomenon. Some basic structural properties of the new distributions like Quantile function, moments, moment generating function, characteristics function and order statistics were obtained [13,14]. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. Behaviour of the hazard rate plot exhibit increase, decrease, Bathtub and inverted Bathtub shape. Maximum likelihood estimate was used to estimate the Z-Kum distribution parameters, Monte Carlo simulation also was carried out to evaluate the performance of MLE in estimating our distribution parameters. Result of the simulation studies revealed that as the sample size increases, the estimate values approaches actual parameter values, while the values of mean square errors approaches zero, this indicates that MLE is good for the estimation of our distribution parameters. To show how flexible and more efficient our proposed model is over some existing distributions, we compare the model with the other fitted existing models. Analytical measure of goodness of fit of some information criterion such as Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) was considered using three real life data sets. From the results obtained, it is evident that our proposed model give better fit than the other competing models and is therefore, more flexible in modeling and predicting real world phenomenon.

Competing Interests

Authors have declared that no competing interests exist.

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