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Evolution of Lovelock Tensor as a Generalized Einstein Tensor and Lovelock Gravity under Ricci Flow

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Abstract

The higher dimensional gravity theory of Lovelock is a fascinating generalization of Einstein's gravity theory and it is of extreme interest in theoretical physics as it delineates a wide class of relativistic models. Here, we propose a short digest on Lovelock theory that represents a very beautiful scenario to study how the differential geometry of gravity results corrected at short distance due to the presence of higher order curvature terms in the action. As in the modern literature of cosmology, the space-time has been supposed to be a dynamical manifold. Hence by admitting this fact in the present study, we will be concerned with the flow equations of all the Lovelock configurations. In particular, we shall make use of Ricci flow techniques to evolve the actions which are responsible for higher order gravity theory. Finally, we shall attempt to evolve the Lovelock tensor to generate a very useful non-linear heat diffusion equation that could analyze the mystery of higher order gravity theory

Keywords: Lovelock; Lagrangian; Dynamical manifold; Ricci Flow (R.F.); Gauss-Bonnet, Einstein
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1 Introduction

Einstein's general theory of relativity (G.R.) has been significantly enhanced and evolved under various advanced differential geometric tools and conceptions and therefore behaving like a matchless extraordinary piece of twentieth century Physics. The Einstein's theory of G.R. includes the rigorous pursuit of the principle of invariance of the laws of physics and therefore leads to a very surprising idea that our space-time is a *dynamical manifold*.
This idea indulged into Physics, an abstract study of non-Euclidean geometry by various Mathematician such as Gauss (1827), Gauss (1965), Gauss (1889), Riemann (2004), Levi-Civita (1899) and Levi-Civita (1927) etc. It also delineates space and time to the status of dynamical structured cosmos. The

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applications of differential geometry to physics also play crucial role in Gauge theory. Moreover, the possibility of representing Weyl's-Gauge (conformal-Gauge) invariant theory in terms of connections of fiber bundles has been of great interest. Afterward, to enhance the study on dynamical structured cosmos, some modifications to Einstein's tensor have been suggested in many contexts like; counter-terms in G.R. to regulate singularities; scalar-tensor theories in inflationary contexts; terms appearing in supergravity; low energy action from strings etc.

Towards the enhancement and reformation of Einstein's tensor for dynamical structured cosmos, various Geometers and Physicists have made extraordinary efforts from their own perspectives. Among them, the first and foremost is Lovelock (1972), who has contributed at large by introducing his very precious master piece *"The four dimensionality of space and Einstein's tensor"* [Lovelock (1972)]. In fact, by considering his work, many have made their fantastic contribution in the modern context of dynamical structured cosmos. Caltenco et al. (2001) in his research work introduced well known Lanczos potential (after the name of Prof. C. Lanczos) for the Kerr metric of dynamical structured cosmos and developed an especial kind of Lanczos spin tensor for a rotating black hole. Further, to pursue electrodynamics of classically charged particles in the dynamical manifold, Arreaga (2000) has proposed some analogies between the Lanczos spin tensor for the conformal tensor of the space-time and the Weert potential for the boundary part of the Liénard-Wiechert electromagnetic field.

The gravity via the equivalence principle has also been a burning topic among all Physicists and Mathematicians, as such kind of conception leads to relative acceleration between local Lorentz frames at different regions. In other sense, space-time is curved and there is no global inertial frame for the distinguished study of G.R. Now, the dynamical manifold being curved, there should be some technique to determine the source of curvature for this . Of course, the matter and non-gravitational energy (in the form of energy-momentum tensor) can be used to meet the purpose. Further, for our dynamical manifold the energy-momentus tensor can be evaluated via famous Einstein equations. To develop such an Einstein's equation, the Mach's principle is customarily used, which states that the definition of inertial frames being dependent on the matter-energy content of the universe. Thus according the Mach's principle, the Einstein's equations are;

$$T_{ij} = G_{ij}, \tag{1.1}$$

or including the cosmological constant Λ, equation (1.1) produces;

$$T_{ij} = G_{ij} + \Lambda g_{ij}, \tag{1.2}$$

where T_{ij} is the energy-momentum tensor and G_{ij} is the Einstein's tensor constructed merely from metric tensor and its derivatives as;

$$G_{ij} = R_{ij} - \frac{R}{2} g_{ij}. \tag{1.3}$$

g_{ij} is the metric tensor which gives a measure of distance, R_{ij} is the contracted curvature or Ricci curvature and is derived from the Riemannian curvature R^h_{ijk} going through usual contraction process. Also, the Riemannian tensor is constructed from the metric compatible, torsion free connection coefficients Γ^i_{jk} and these connections are often called Levi-Civita connection coefficients, or sometimes the Christoffel's three index symbols [Kreyszig (1991)]. The metric compatible connection defines the parallel transport of vectors such that their metric products (norm and angle) are preserved. Moreover, the Riemannian tensor measures the non-commutativity of the associated covariant derivative. The symbol $R = R^i_i$ is the Ricci scalar.

Now, the Einstein's equations with cosmological constant obey three very important principles [Zanelli (2001-02)]:

(a). They are independent of reference frame determined by choice of co-ordinates.

(b). There is well defined Cauchy problem for the evolution of the metric tensor [Wald (1984)].

(c). In the non-relativistic case, they reduce to Newtonian Gravity in weak field.

Here the condition (a) is naturally satisfied because of Einstein's equations composed of tensorial quantities. Condition (b) implies that the Cauchy's conditions are necessary and sufficient to integrate vacuum Einstein's equations specifying the field and its first derivatives on an initial Cauchy surface. The Cauchy's conditions can be done by means of Hamiltonian formulation which evolves the field (gravitational or electromagnetic) with respect to time. The form of Einstein's tensor ensures that the metric has a well defined Cauchy problem and therefore, the whole Einstein's equations have such a well defined initial value problem , which depend on the matter content. Eventually, the condition (c) is exactly in accordance with the measurement from everyday physics to celestial mechanics that all non-relativistic, classical weak gravity systems obey Newtonian physics.

In case, if one insists that the dynamical structured cosmos has a vacuum solution, then one should eliminate the cosmological term or assume it very very small. Also, Einstein's tensor bears some significant features which assign it very suitability to generalize field equations for dynamical manifold. These features are:
(I). Symmetry. (II). Covariantly conserved, i.e., $\nabla_j G^i_k = 0$. (III). Depending only on metric, its first and second derivatives. (IV). Linear in second derivative of the metric.
The importance of features (I) and (II) is apparent since an energy-momentum tensor comes from the variation of a matter Lagrangian with respect to the metric, is both symmetric and divergence free. The action which produces the Einstein's field equations is known as Einstein-Hilbert action defined as:

$$S = \int_M \sqrt{(g)} R.d^4x, \tag{1.4}$$

where $dv = \sqrt{(g)}dx^0dx^1dx^2dx^3 = \sqrt{(g)}d^4x$ defines the elementary four dimensional volume of dynamical manifold and $g \equiv det\ g_{ij}$.
Moreover, the feature (II), the contracted Bianchi's identity gives the local conservation of energy-momentum. The features (III) and (IV) are significant for the physical conditions (b) and (c).

We, now, proceed to outline some more basic but complicated formalism regarding theory of gravitation including the most desiring Lovelock gravity.

1.1 Gravitational Field Theory from Lovelock Perspective

The theory of gravitational field based on Lagrangian quadratic in the curvature tensor has been introduced by Eddington (1975), Weyl (1918), Weyl (1919), Weyl (1918-21) and Weyl (1921). Perhaps, a Mathematical inspiration to examine gravitational theories fabricated on non-linear Lagrangian has been the phenomenological aspect of Einstein's theory. This means that there is a direct dependence of Einstein's tensor and Lagrangian on the derivatives of metric [Farhoudi (1995)]. Recently, it has been well known to us that the Einstein's gravity, which when treated under fundamental quantum gravity, leads to a non-renormalizable theory. Therefore, in order to permit renormalization of the divergence, the quantum gravity indicates that the Einstein's-Hilbert action should be extended by the insertion of higher order gravity terms [Utiyama and DeWitt (1962)]. Moreover, certain theories of gravity with curvature tensor squared terms have been suggested to render gravity renormalizable in four dimensions and in fact it has been shown [Birrell and Davies (1982), Buchbinder et al. (1992)] that the Lagrangian ;

$$L = \frac{1}{k^2}\left(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right), \tag{1.5}$$

which due to Gauss Bonnet theorem, is the most general quadratic Lagrangian upto four dimensions used to solve renormalizable problems. Here α, β and $k^2 \equiv \frac{16\pi G}{c^4}$ are constants.
The above Lagrangian is multiplicatively renormalizable [Stelle (1977)] and asymptotically free [Fradkin and Tseytalin (1982)]; however, it will not be unitary in the sense of scalar product or norm invariance. It is also discussed by Farhoudi (2009) that actually the above Lagrangian within usual perturbation theory has a particle spectrum containing a further massive scalar spin two ghost which has either negative energy or a negative norm and the existence of negative excitations in a relativistic model leads to causality violation [Stelle (1978)].
This is the main reason that lack of unitarity removes the possibility of higher order gravity inclusion in

the above Lagrangian. Further, Zwiebach (1985) and Zumino (1986) have shown that the quadratic Lagrangian by means of dimensionally continued Gauss-Bonnet densities must have the form of Lovelock Lagrangian [Lovelock (1973), Lovelock (1972), Briggs (1998)] as;

$$\pounds = \frac{1}{k^2} \sum_{0<n<\frac{D}{2}} \frac{1}{2^n} c_n \delta^{\alpha_1 \cdots \alpha_{2n}}_{\beta_1 \cdots \beta_{2n}} R^{\beta_1 \beta_2}_{\alpha_1 \alpha_2} \cdots R^{\beta_{2n-1} \beta_{2n}}_{\alpha_{2n-1} \alpha_{2n}} \equiv \sum_{0<n<\frac{D}{2}} c_n L^{(n)}, \tag{1.6}$$

where $c_1 \equiv 1$ is set and other constants c_n are taken to be of the order of Plank's length to the power $2(n-1)$ for the dimension of $\pounds$ to be same as $L^{(1)}$. Also, the symbol $\delta^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_p}$ is the totally antisymmetric Kronecker delta, which is identically zero if $p > D$ and the supremum value of p is concerned to the dimension D of structured cosmos having conformal configurations and is defined by;

$$n_{sup} = \begin{cases} \frac{D}{2} - 1, & even\ D \\ \frac{D-1}{2}, & odd\ D \end{cases}. \tag{1.7}$$

Now, this above ghost free feature including the fact that the Lovelock Lagrangian in the most general second order Lagrangian which is same as the Einstein-Hilbert Lagrangian that produces the field equation as second order equation and such a feature stimulates importance of Lovelock gravity and its significance in the literature of theory of relativity [Farhoudi (2006), Nojiri and Odintsov (2007)]. Moreover, the Lovelock Lagrangian evidently reduces to Einstein-Hilbert Lagrangian and its second term is the Gauss-Bonnet's invariant given by;

$$L^{(2)} = \frac{1}{k^2} \left(R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right). \tag{1.8}$$

Lovelock (1973) and Briggs (1998) have also discussed that each term of the Lovelock tensor $\mathbb{G}_{\alpha\beta}$, where

$$\mathbb{G}_{\alpha\beta} = - \sum_{0<n<\frac{D}{2}} \frac{1}{2^{n+1}} c_n g_{\alpha\mu} \delta^{\mu\alpha_1 \cdots \alpha_{2n}}_{\beta\beta_1 \cdots \beta_{2n}} R^{\beta_1 \beta_2}_{\alpha_1 \alpha_2} \cdots R^{\beta_{2n-1} \beta_{2n}}_{\alpha_{2n-1} \alpha_{2n}} \equiv \sum_{0<n<\frac{D}{2}} c_n G^{(n)}_{\alpha\beta}, \tag{1.9}$$

has some remarkable features. That is to say, each term of $G^{(n)}_{\alpha\beta}$ can be re-written in the form of Einstein's tensor with respect to Ricci and scalar curvature tensor as follows:

$$G^{(n)}_{\alpha\beta} = R^{(n)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(n)}, \tag{1.10}$$

here the Lovelock Lagrangian configurations, i.e., $R^{(n)}_{\alpha\beta}$ and $R^{(n)}$ are defined as;

$$R^{(n)}_{\alpha\beta} \equiv \frac{n}{2^n} \delta^{\alpha_1 \alpha_2 \cdots \alpha_{2n}}_{\alpha\beta_2 \cdots \beta_{2n}} R^{\beta_2}_{\alpha_1 \alpha_2 \beta} R^{\beta_3 \beta_4}_{\alpha_3 \alpha_4} \cdots R^{\beta_{2n-1} \beta_{2n}}_{\alpha_{2n-1} \alpha_{2n}} \tag{1.11}$$

and

$$R^{(n)} \equiv \frac{1}{2^n} \delta^{\alpha_1 \alpha_2 \cdots \alpha_{2n}}_{\beta_1 \beta_2 \cdots \beta_{2n}} R^{\beta_1 \beta_2}_{\alpha_1 \alpha_2} \cdots R^{\beta_{2n-1} \beta_{2n}}_{\alpha_{2n-1} \alpha_{2n}}. \tag{1.12}$$

Also, from these equations, one can easily observe that $R^{(1)}_{\alpha\beta} \equiv R_{\alpha\beta}$ and $R^{(1)} \equiv R$.

The Euler-Lagrangian configurations, i.e., $R^{(1)}_{\alpha\beta}$ and $R^{(1)}$ can be obtained easily by using definition of generalized Kronecker delta and the properties of Riemannian-Christoffel's tensor. Farhoudi (2009) has also delineated an alternative and very basic technique to notice that under the process of varying Einstein-Hilbert action; $\delta \int L^{(n)} \sqrt{(-g)}\, d^D x$ along with its Euler-Lagrangian expression;

$$\frac{\delta L^{(n)}}{\delta g^{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} L^{(n)} \equiv \frac{1}{k^2} G^{(n)}_{\alpha\beta}, \tag{1.13}$$

the relations $(6), (11)$ and (12) produce

$$\frac{\delta L^{(n)}}{\delta g^{\alpha\beta}} = \frac{1}{k^2} R^{(n)}_{\alpha\beta} \text{ and } L^{(n)} = \frac{1}{k^2} R^{(n)}. \tag{1.14}$$

The above expression is very much straightforward but there exists a relation between $R^{(n)}_{\alpha\beta}$ and $R^{(n)}$ analogous to that which exists between Ricci tensor and scalar curvature tensor; namely

$$\frac{1}{n}\,\mathrm{trace}\,R^{(n)}_{\alpha\beta} = R^{(n)}, \tag{1.15}$$

here the "trace" stands for usual contraction of any two indices of the corresponding tensorial quantity, for instance, $\mathrm{trace}\,A_{\mu\nu} \equiv g^{\alpha\beta}A_{\alpha\beta}$.
Thereby, from the above detail, one can conclude that the splitting or decomposing feature of Einstein's tensor, as a first terms of Lovelock tensor into two parts with the aforementioned trace relation between them is a common feature of any other part of Lovelock tensor, in which each term is merely a homogeneous Lagrangian. Now, it is also a point to be taken under trial that from the standpoint of principle of invariance, what might happen with Lagrangian? Usually, its relevant Euler-Lagrangian expression becomes inhomogeneous tensor as for instance, the entire Lovelock Lagrangian $\pounds$, which is compound of terms with a mixture of different orders.
Thus in case of inhomogeneity, the relevant Euler-Lagrangian can easily be mentioned by analogy with the form of $G^{(n)}_{\alpha\beta}$, for instance;

$$\mathbb{G}_{\alpha\beta} = \mathbb{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathbb{R},$$

where

$$\mathbb{R}_{\alpha\beta} \equiv \sum_{0<n<\frac{D}{2}} c_n R^{(n)}_{\alpha\beta} \text{ and } \mathbb{R} \equiv c_n R^{(n)}. \tag{1.16}$$

Farhoudi (2009) also discussed that a similar relation like (1.15) can not be setup for (1.16) due to the involvement of the factor $\frac{1}{n}$. Hence to overcome this task, Farhoudi (2009) has introduced a new avenue of generalized trace tool as an extra mathematical formulation for Riemannian framework. This tool may slightly deform the original form of trace relation and modify it adequately, to enable one to deal with difficulty of inhomogeneity. Here is the concise discussion of Farhoudi (2009) over generalized trace technique.

1.2 Generalized Trace Technique for Inhomogeneous Lovelock Tensor

In the present section, Farhoudi (2009) attempted to delineate generalized trace used to solve the difficulty in case of inhomogeneous Lovelock tensor, whose components are homogeneous functions of the metric tensor and its derivatives.
To develop the generalized trace technique, Farhoudi (2009) has mentioned that either $g_{\mu\nu}$ or $g^{\mu\nu}$ should be selected as a base for enumerating the homogeneity degree number. Further, he choose the homogeneity degree number (here and hereafter abbreviated as "HDN") of $g^{\mu\nu}$ as $[+1]$; hence the HDN of $g_{\mu\nu}$ as $[-1]$, since $g^{\mu\nu}g_{\mu\alpha} = \delta^{\nu}_{\alpha}$. Hence as the contravariant and covariant tensors are transformed into each other in a $1-1$ pattern by the metric, their HDNs are differed by $[\pm 2]$. Similarly, one can choose the HDN of $g^{\mu\nu}_{,\alpha}$ as $[+1]$ and therefore the HDN of $g_{\mu\nu,\alpha}$ will be $[-1]$ and would be calculated from $g_{\mu\nu,\alpha} = -g_{\mu\theta}g_{\nu\beta}g^{\theta\beta}_{,\alpha}$. Moreover, to identify the HDNs of higher order derivatives of the metric, one may assume ∂_α as of the HDN zero and thereby the HDN of $\partial^\alpha = g^{\alpha\beta}\partial_\beta$ will be $[+1]$. This all is due to elementary property of homogeneous function that the HDN of a term composed of cross functions can be obtained by adding the HDNs of each of the cross functions. Farhoudi (2009) has calculated the HNDs (say h) for some important homogeneous functions of the metric and its derivatives. We, now, mention the generalized trace formula for a general (p,q) type tensor, e.g., $A^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q}$, which is a homogeneous function of degree h with respect to metric and its derivatives.

We shall consult table (1) of Farhoudi (2009) for HDNs and prefer to use notations as mentioned by Farhoudi (2009)

The generalized trace for such a homogeneous function has been defined by Farhoudi (2009) as:

$$\text{Trace }^{[h]}A^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q} = \begin{cases} \frac{1}{h-\frac{p}{2}+\frac{q}{2}} \text{ trace }^{[h]}A^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q}, & \text{when } h-\frac{p}{2}+\frac{q}{2}\neq 0 \\ \text{trace }^{[h]}A^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q}, & \text{when } h-\frac{p}{2}+\frac{q}{2}=0. \end{cases} \quad (1.17)$$

It is evident from the above relation that contravariant and covariant components of a tensor have different HDNs , however the equality of traces is still retained, for instance;
$\text{Trace } A_{mu\nu} = \text{ Trace } A^{\mu\nu} \equiv A$, whatever be the HDNs of them, just like $\text{trace } A_{mu\nu} = \text{ trace } A^{\mu\nu} \equiv A^{\alpha}_{\alpha}$. Then from (1.17), it follows that;

$$\text{Trace }^{[h]}A_{\mu\nu} = \begin{cases} A = \frac{1}{h+1}A^{\alpha}_{\alpha}, & \text{for } h\neq -1 \\ A = A^{\alpha}_{\alpha}, & \text{for } h=-1. \end{cases} \quad (1.18)$$

In practice, the generalized trace by definition has all properties of usual trace, for instance its invariance under a similar transformation (i.e., for similar tensors), if the transformation doesn't change the HDNs of the tensor and its basis independence for linear operation in finite dimensional Hilbert space. However, the trace operator can not act as a linear operator, when the coefficients of linearity themselves are the homogeneous functions of degree $h' \neq 0$. Using the definition of trace, for the case when $h' \neq 0$, we have, for instance

$$\begin{aligned} \text{Trace } \left({}^{[h']}C^{[h]}A_{\mu\nu}\right) &= \frac{1}{h'+h+1} \text{ trace } \left({}^{[h']}C^{[h]}A_{\mu\nu}\right), \text{ for } h'+h\neq -1 \\ &= \frac{{}^{[h']}C}{h'+h+1} \text{ trace }^{[h]}A_{\mu\nu}. \end{aligned}$$

Once more, using the definition of generalized space, we get

$$\text{Trace } \left({}^{[h']}C^{[h]}A_{\mu\nu}\right) = \begin{cases} \frac{h+1}{h'+h+1}{}^{[h']}C \text{ Trace }^{[h]}A_{\mu\nu}, & \text{for } h\neq -1 \\ \frac{1}{h'}{}^{[h']}C \text{ Trace }^{[h]}A_{\mu\nu}, & \text{for } h=-1. \end{cases} \quad (1.19)$$

Besides this, we can write

$$\text{Trace } \left({}^{[h']}C^{[h]}A_{\mu\nu}\right) = \begin{cases} {}^{[h']}C \text{ trace }^{[h]}A_{\mu\nu}, & \text{for } h'+h=-1 \\ (h+1)^{[h']}C \text{ Trace}^{[h]}A_{\mu\nu}, & \text{for } h\neq -1. \end{cases} \quad (1.20)$$

Because of the involvement of extra factor $(h+1)$ in equation (1.20) and $\frac{h+1}{h'+h+1}$ in equation (1.19), the Trace can not be assumed as a linear operator. However, these extra functions can be made equal to one, only when $h=-1$ and $h'=+1$, or when $h=0$ and $h'=-1$ in (1.19) and (1.20) respectively. Farhoudi (2009) has mentioned that to define generalized trace for inhomogeneous function, one should emphasize on the distributivity of usual trace, which can happen in the case when there are either no coefficients of linearity, or when coefficients are included with their associated tensor, and or when coefficients are assumed to be scalar with $h'=0$.
Here, for dealing with inhomogeneous Lagrangian, it can be shown that the definition of Trace also has a link with Euler's theorem for homogeneous functions. The Euler's theorem for homogeneous functions states that:

Theorem 1.1. *"if $A(g^{\mu\nu})$ is a homogeneous scalar function of degree $[h]$, i.e., $A(\lambda g^{\mu\nu}) = \lambda^h A(g^{\mu\nu})$, then $g^{\mu\nu}\frac{\partial A}{\partial g^{\mu\nu}} = hA$". Where we formally define $\frac{\partial A}{\partial g^{\mu\nu}} \equiv A_{\mu\nu}$ and $A_{\mu\nu}$ is of degree $[h-1]$. Also, $g^{\mu\nu}A_{\mu\nu}$ denotes its usual trace defined by $A \equiv \text{Trace } A_{\mu\nu}$.*

Hence from Euler's theorem, one is able to derive the trace of a tensor as;

$$\text{trace }^{[h-1]}A_{\mu\nu} = h \text{ Trace } A_{\mu\nu},$$

or,

$$\text{Trace}\ {}^{[h-1]}A_{\mu\nu} = \frac{1}{h}\ \text{trace}\ A_{\mu\nu},\ \text{when}\ h \neq 0. \tag{1.21}$$

Furthermore, under the generalized trace (1.17), one has

$$\text{Trace}\ {}^{[+1]}g^{\mu\nu} = \ \text{trace}\ g^{\mu\nu} = D$$

and

$$\text{Trace}\ {}^{[-1]}g_{\mu\nu} = \ \text{trace}\ g_{\mu\nu} = D.$$

Also, here is an example given by Lovelock (1971), when the definition of generalized trace is applied to equation (1.11), an expression similar to equation (1.15) can be obtained including more analogous form for each other, namely;

$$\left.\begin{aligned} \text{Trace}\ R^{(1)}_{\alpha\beta} &= R^{(1)} = k^2 L^{(1)} \\ \text{Trace}\ R^{(2)}_{\alpha\beta} &= R^{(2)} = k^2 L^{(2)} \\ \text{Trace}\ R^{(3)}_{\alpha\beta} &= R^{(3)} = k^2 L^{(3)} \\ &\vdots \\ \text{Trace}\ R^{(n)}_{\alpha\beta} &= R^{(n)} = k^2 L^{(n)} \end{aligned}\right\}, \tag{1.22}$$

where from equation (1.18), we evidently have

$$R^{(n)} = \frac{1}{n} R^{(n)\rho}_{\rho} \tag{1.23}$$

It has been also delineated that in case of inhomogeneous Lovelock Lagrangian, under the efficiency of generalized trace operator, the Lovelock tensor can be written as;

$$\mathbb{G}_{\alpha\beta} = \mathbb{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathbb{R}. \tag{1.24}$$

Moreover, by placing for $R^{(n)}$ from (1.12) and using (1.6), we get

$$\pounds = \frac{1}{k^2}\mathbb{R}. \tag{1.25}$$

Also, by substituting it from (1.22) and using distributivity of Trace, we obtain

$$\mathbb{R} = \ \text{Trace}\ \mathbb{R}_{\alpha\beta}. \tag{1.26}$$

Hence, in higher dimensional dynamical structured cosmos, the Lovelock tensor reduces to the Einstein's tensor in four dimensions. More precisely, in higher order gravities, where the geometry is represented by the Lovelock tensor, the field equations can be written as;

$$\mathbb{G} = \frac{1}{2} k^2 T_{\alpha\beta}. \tag{1.27}$$

The Lovelock tensor, now can be classified as a generalized Einstein's tensor and we evoke $\pounds$, $\mathbb{R}_{\alpha\beta}$ and $\mathbb{R}$ the generalized Einstein's gravitational Lagrangian, the generalized Ricci tensor and generalized curvature scalar, respectively.

2 Evolution of Lovelock Configurations Under Ricci Flow Mechanism

Before evolving the foregoing Lovelock configurations under the Ricci flow mechanism, let us launch a brief digest over Ricci flow mechanism.

The Ricci flow (in abbreviated form "R.F.") is an evolution system on metrics and is a mean by which one can take an arbitrary Riemannian manifold and smooth out geometry of that manifold to make it look more symmetric. For a given metric as an initial data, its local existence and uniqueness on the compact manifold was first established by Sir Hamilton. The R.F., which evolves a Riemannian metric by its Ricci curvature, is a natural analogy of heat diffusion equation for metrics. As a consequence, the curvature tensor evolved by a system of diffusion reaction equations which tend to distribute the curvatures uniformly over the manifold. Hence, one expects that the initial metric be improved and evolved into a more canonical metric, thereby leading to a better understanding of the Topology of the underlying manifold.
Now, as it has already been mentioned that the Einstein's theory of G.R. is recently attracting towards a very surprising notion of "dynamical manifold". Therefore to reform our pursuance concerning evolution of Lovelock configurations of such a dynamical structured cosmos, we outline some appropriate R.F. techniques which would cooperate us to disclose few more mysteries of space-time.
We assume that our Einstein manifold EM bears all the features of dynamical structured manifold, i.e., all the Einstein's geometric configurations have varying nature and thus obviously have the dependence upon time factor. Moreover, under this assumption, naturally the Einstein's metric g_{ij} will depend upon time factor. Therfore, while applying R.F. mechanism for such a metric, one is desperate to know, how the various other configurations evolve?
Suppose $g_{ij}(t)$ is a time-dependent Einstein metric and

$$\frac{\partial}{\partial t}g_{ij}(t) = h_{ij}(t), \tag{2.1}$$

where h_{ij} is some symmetric 2-tensor.
Then under the above time evolving metric, the various geometric quantities evolve according to the following expressions [Sheridan (2006)]:

$$\frac{\partial}{\partial t}g^{ij} = -h^{ij} = -g^{ik}g^{jl}h_{kl}, \tag{2.2}$$

where g^{ij} is th inverse of Einstein metric tensor.

$$\text{Christoffel symbol: } \frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}). \tag{2.3}$$

$$\text{Riemannian tensor: } \frac{\partial}{\partial t}R^l_{ijk} = \frac{1}{2}g^{lp}[\nabla_i\nabla_j h_{kp}+\nabla_i\nabla_k h_{jp}-\nabla_i\nabla_p h_{jk}-\nabla_j\nabla_i h_{kp}-\nabla_j\nabla_k h_{ip}+\nabla_j\nabla_p h_{ik}]. \tag{2.4}$$

$$\text{Ricci tensor: } \frac{\partial}{\partial t}R_{ij} = \frac{1}{2}g^{pq}(\nabla_q\nabla_i h_{jp} + \nabla_q\nabla_j h_{ip} - \nabla_q\nabla_p h_{ij} - \nabla_i\nabla_j h_{qp}). \tag{2.5}$$

$$\text{Scalar tensor:} \frac{\partial}{\partial t}R = -\triangle H + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq}, \tag{2.6}$$

where $\triangle = \nabla_p\nabla^p$ is the Laplacian and $H = g^{pq}h_{pq}$.

$$\text{Volume element: } \frac{\partial}{\partial t}d\mu = \frac{H}{2}d\mu, \tag{2.7}$$

where $d\mu = \sqrt{(\text{det}g_{ij})}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

$$\text{Volume of Einstein manifold: } \frac{d}{dt}\int_{EM} d\mu = \int_{EM}\frac{H}{2}d\mu. \tag{2.8}$$

$$\text{Total curvature scalar on a closed EM: } \frac{d}{dt}\int_{EM} Rd\mu = \int_{EM}(\frac{1}{2}RH - h^{ij}R_{ij})d\mu. \tag{2.9}$$

Besides the above non-linear time dependent heat diffusion expressions, we delineate some more typical time dependent non-linear heat diffusion expressions, which significantly participate in the

Lovelock theory of gravity.
For any time dependent tensor $\alpha \in EM$, there holds

$$\frac{\partial}{\partial t}(\text{trace } \alpha) = - < h, \alpha > + \text{trace}\frac{\partial \alpha}{\partial t}, \tag{2.10}$$

where the symbol $<, >$ stands for the inner product.
The Ricci tensor, under the concept of divergence and Lichnerowicz Laplacian evolves as;

$$\frac{\partial}{\partial t}\text{Ric} = -\frac{1}{2}\triangle_L h - \frac{1}{2}\text{Ł}_{\delta^*(G(h))}. \tag{2.11}$$

Here, δ^* is the divergence operator such that $\delta^* : EM \rightarrow EM_{hypersurface}$ and for various T it is defined as $\delta^*(T) = -\text{trace}_{12}\nabla T$. Again, $\delta^*(T) = -\text{trace}_{12}\nabla T$ means the trace taken over first and second entries. The symbol Ł used in equation (2.11) stands for the Perelman's length. The symbol $\triangle_L$ is the Lichnerowicz Laplacian and will be defined later. Also, in the present study, we have assumed an Einstein's metric $g(t)$ depending upon the time factor such that $t \in [0, T]$.
Moreover, some various $t \in [0, T] \in EM$, we have the following useful identities:

$$\text{The Gravitational tensor : } G(T) = T - \frac{1}{2}(\text{trace } T)g. \tag{2.12}$$

Tanking the divergence of (2.12), we have,

$$\delta^* G(T) = \delta^* T + \frac{1}{2}d(\text{trace } T). \tag{2.13}$$

In view of equation (2.13), we have an important identity;

$$\delta^* G(\text{Ric}) = \delta^* \text{ Ric} + \frac{1}{2}dR = 0. \tag{2.14}$$

We now describe the Lichnerowicz Laplacian involved in equation (2.11). This operator in global co-ordinate system is given as;

$$(\triangle_L h)(X, W) = (\triangle h)(X, W) + h(X, \text{Ric}(W)) - h(W, \text{Ric}(W)) - h(W, (\text{Ric}(X))) + 2\text{trace } h(R(X, .)W, .), \tag{2.15}$$

while the same operator in local coordinate system yields;

$$(\triangle_L h)_{ij} = \triangle h_{ij} + 2R_{kijl}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki}. \tag{2.16}$$

Besides, if one prompts himself to go through the linearized Ricci flow system, then one needs to think about a solution of linearized R.F. system composed of a complete solution $(EM^4, g_0(t)), t \in [0, T]$, to the R.F. :

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{2.17}$$

coupled with a solution $h(t), t \in [0, T]$, to Lichnerowicz Laplacian heat diffusion equation:

$$\frac{\partial}{\partial t}h_{ij} = (\triangle_L h)_{ij}\,. \tag{2.18}$$

Once again, if we write $\alpha = \alpha_{ij}dx^i \otimes dx^j$ and taking care of equations (2.1) and (2.2), we have;

$$\frac{\partial}{\partial t}(\text{trace } \alpha) = \frac{\partial}{\partial t}\left(g^{ij}\alpha_{ij}\right) = -h^{ij}\alpha_{ij} + g^{ij}\frac{\partial}{\partial t}\alpha_{ij} =< h, \alpha > + \text{trace } \frac{\partial \alpha}{\partial t}. \tag{2.19}$$

From which, we can have an expression of the form:

$$\frac{\partial R}{\partial t} \equiv \frac{\partial}{\partial t}(\text{trace Ric}) = - < h, \text{ Ric} > + \text{trace } (\frac{\partial}{\partial t}\text{ Ric}). \tag{2.20}$$

Now, as the higher dimensional Lovelock theory of gravity is an interesting generalization of Einstein's G.R. theory. Therefore, from the standpoint of very crucial fact that space-time is a dynamical manifold, we draw our focus towards the evolution of Lovelock configurations under the aforementioned R.F. mechanism.
By evolving the Lovelock configurations, we shall try to produce some useful non-linear heat diffusion equations, which would be very much helpful in knowing the varying nature of the underlying dynamical manifold EM.
Let us first evolve the Einstein's equation following the well known Mach's principle under suitable R.F. mechanism discussed in equations (2.1) to (2.20). Suppose the energy momentum tensor T_{ij} is the time dependent configuration of EM-manifold, such that;

$$T_{ij}(t) = G_{ij}(t) + \Lambda g_{ij}(t), \forall t \in [0, T]. \tag{2.21}$$

Applying R.F. on both side of equation (2.21), it follows that

$$\frac{\partial}{\partial t}T_{ij}(t) = \frac{\partial}{\partial t}[G_{ij}(t) + \Lambda g_{ih}(t)], \tag{2.22}$$

where $G_{ij}(t)$ is the time dependent Einstein's tensor given by equation (1.3).
After some straightforward and simple calculations, equation (2.22) yields,

$$\frac{\partial}{\partial t}T_{ij}(t) = \frac{1}{2}g^{pq}(\nabla_q\nabla_i h_{jp} + \nabla_q\nabla_j h_{ip} - \nabla_q\nabla_p h_{ij} - \nabla_i\nabla_j h_{qp}) + \\ + \frac{1}{2}g_{ij}(\triangle H - \nabla^p\nabla^q h_{pq} + h^{pq}R_{pq}) - \frac{R}{2}h_{ij} + \Lambda h_{ij}. \tag{2.23}$$

In addition to warm up the Lovelock literature, we actually started with a very simple Einstein-Hilbert action (1.4), which in generalized form is rewritten as;

$$S = \frac{1}{16\pi G}\int dx^4\sqrt{-g}(R - 2\Lambda + \alpha R^2),$$

or, in more concise form written as

$$S = \frac{1}{16\pi G}\int_{EM} d\mu(R - 2\Lambda + \alpha R^2). \tag{2.24}$$

The equation (2.24) corresponds to Einstein-Hilbert action in four dimensions augmented with the square of curvature scalar, where α is a coupling constant with dimensions of $[\alpha] =$ length^2. This action is a particular case of the so called $f(R)$-gravity theories which are defined by adding to the Einstein-Hilbert Lagrangian a function of the Ricci scalar $f(R)$. But the theory defined by equation (2.24) is not the only theory of gravity that admits Schwarzchild metric as a persistent solution.
Actually, this is a rather common feature of theories with higher curvature terms. Besides this, we can consider another action that introduces Einstein's gravity coupled to conformally invariant gravity, namely

$$S = \frac{1}{16\pi G}\int_{EM} d\mu(R - 2\Lambda + c\, W_{\alpha\beta\theta\psi}W^{\alpha\beta\theta\psi}), \tag{2.25}$$

where c is a coupling constant and $W_{\alpha\beta\theta\psi}$ is the Weyl tensor, whose quadratic contraction is delineated as;

$$W_{\alpha\beta\theta\psi}W^{\alpha\beta\theta\psi} = \frac{1}{3}R^2 - 2R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\theta\psi}R^{\alpha\beta\theta\psi}. \tag{2.26}$$

We, now, compute the non-linear heat diffusion equation for the Einstein-Hilbert actions (2.24) and (2.26) under the suitable R.F. techniques as follows:
Applying the R.F. on both side of action (2.24) and taking care of equations (2.6) and (2.9), we have

$$\frac{\partial S}{\partial t} = \frac{1}{16\pi G}\frac{d}{dt}\int_{EM} R d\mu - \frac{\Lambda}{8\pi G}\frac{d}{dt}\int_{EM} d\mu + \frac{\alpha}{16\pi G}\frac{d}{dt}\int_{EM} R^2.d\mu.$$

The above expression after some manipulation turns out to be a required non-linear heat equation as;

$$\frac{\partial S}{\partial t} = \frac{1}{16\pi G}\int_{EM}(\frac{1}{2}RH - h^{ij}R_{ij})d\mu - \frac{\Lambda}{8\pi G}\int_{EM}\frac{H}{2}d\mu + \\ + \frac{\alpha}{16\pi G}\int_{EM}[(-\triangle H + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq}) + (\frac{1}{2}RH - h^{ij}R_{ij})]R.d\mu. \quad (2.27)$$

Further, to evolve Einstein field equations, applying R.F. mechanism to Einstein-Hilbert action (2.25), we have

$$\frac{\partial S}{\partial t} = \frac{1}{16\pi G}\frac{d}{dt}\int_{EM}Rd\mu - \frac{\Lambda}{8\pi G}\frac{d}{dt}\int_{EM}d\mu + \frac{c}{48\pi G}\int_{EM}R^2 d\mu - \frac{c}{8\pi G}\frac{d}{dt}\int_{EM}(R_{\alpha\beta}R^{\alpha\beta}d\mu) + \\ + \frac{c}{16\pi G}\frac{d}{dt}\int_{EM}(R_{\alpha\beta\theta\psi}R^{\alpha\beta\theta\psi}d\mu).$$

Manipulating the above expression by employing equations (2.4), (2.5), (2.6), (2.8) and (2.9), we have after some lengthy but straightforward computations;

$$\frac{\partial S}{\partial t} = \frac{1}{16\pi G}\int_{EM}(\frac{1}{2}RH - h^{ij}R_{ij})d\mu - \frac{\Lambda}{8\pi G}\int_{EM}\frac{H}{2}d\mu + \frac{c}{48\pi G}\int_{EM}[(-\triangle H + \\ \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq}) + +(\frac{1}{2}RH - h^{ij}R_{ij})]R.d\mu - \frac{c}{8\pi G}\int_{EM}[R_{\alpha\beta}\{d\mu(\frac{1}{2}g^{pq}(\nabla_q\nabla^\alpha h_p^\beta + \\ \nabla_q\nabla^\beta h_p^\alpha - \nabla_q\nabla_p h^{\alpha\beta}h_{pq})) + +R^{\alpha\beta}\frac{H}{2}d\mu\} + R^{\alpha\beta}d\mu \times \frac{1}{2}g^{pq}(\nabla_q\nabla_i h_{jp} + \nabla_q\nabla_j h_{ip} - \nabla_i\nabla_j h_{qp})] + \\ \frac{c}{16\pi G}\int_{EM}[R_{\alpha\beta\theta\psi}\{d\mu\frac{1}{2}g^{\alpha p}(\nabla^\theta\nabla^\psi h_p^\beta + +\nabla^\psi\nabla^\beta h_p^\theta - \nabla^\psi\nabla_p h^{\beta\theta} - \\ \nabla^\psi\nabla^\theta h_p^\beta - \nabla^\theta\nabla^\beta h_p^\psi + \nabla^\theta\nabla_p h^{\beta\psi})\} + R^{\alpha\beta\theta\psi}d\mu \times \frac{1}{2}(\nabla_\alpha\nabla_\beta h_{\theta\psi} + \nabla_\alpha\nabla_\theta h_{\beta\psi} - \\ - \nabla_\alpha\nabla_\psi h_{\beta\theta} - \nabla_\beta\nabla_\alpha h_{\theta\psi} - \nabla_\beta\nabla_\theta h_{\alpha\psi} + \nabla_\beta\nabla_\psi h_{\alpha\theta})]. \quad (2.28)$$

Now, as in subsection (1.1), we have discussed the gravitational field theory from Lovelock perspective and observed that the non-linear Lagrangian has been a crucial aspect of Einstein's theory. Thus, having the dependence of Einstein's tensor and Lagrangian on the derivatives of metric, we evolve the most general quadratic Lagrangian (1.5) so that it could evoke about the gravitational waves for dynamical manifold EM when treated under R.F. mechanism.
Taking R.F. of (1.5) and evolving under appropriate R.F. techniques, after some careful computations, we have

$$\frac{\partial L}{\partial t} = \frac{1}{k^2}[\frac{\partial}{\partial t}R + \alpha\frac{\partial}{\partial t}R^2 + \beta\frac{\partial}{\partial t}(R_{\mu\nu}R^{\mu\nu})] = \frac{1}{k^2}[(-\triangle H + \nabla^\nu\nabla^\mu h_{\nu\mu} - h^{\nu\mu}R_{\nu\mu}) + \\ + 2\alpha R(-\triangle H + \nabla^\nu\nabla^\mu h_{\nu\mu} - h^{\nu\mu}R_{\nu\mu}) + \beta(\frac{1}{2}R_{\mu\nu}g^{pq}\{\nabla_q\nabla^\mu h_p^\nu + \nabla_q\nabla^\nu h_p^\mu - \nabla_q\nabla_p h^{\mu\nu}h_{pq}\} + \\ + \frac{1}{2}R^{\mu\nu}g^{pq}\{\nabla_q\nabla_\mu h_{\nu p} + \nabla_q\nabla_\nu h_{\mu p} - \nabla_q\nabla_p h_{\mu\nu} - \nabla_\mu\nabla_\nu h_{pq}\})]. \quad (2.29)$$

Also, the ghost free feature of Lovelock Lagrangian stimulates that this kind of configuration is the Einstein-Hilbert Lagrangian of second order (1.8), that produces the field equation. Thereby, the heat equation for such configuration in the dynamical manifold in terms of Gauss-Bonnet's densities is given by the following expression;

$$\frac{\partial}{\partial t}L^{(2)}=\frac{1}{k^2}[2R(-\triangle H+\nabla^{\nu}\nabla^{\mu}h_{\mu\nu}-h^{\mu\nu}R_{\mu\nu})-4\{\frac{1}{2}R_{\mu\nu}g^{pq}(\nabla_q\nabla^{\mu}h^{\nu}_p+\nabla_q\nabla^{\nu}h^{\mu}_p-\nabla_q\nabla_p h^{\mu\nu}h_{pq})+ +\frac{1}{2}R^{\mu\nu}g^{pq}(\nabla_q\nabla_m uh_{\nu p}+\nabla_q\nabla_n uh_{\mu p}-\nabla_q\nabla_p h_{\mu\nu}-\nabla_\mu\nabla_\nu h_{pq})\}+R_{\alpha\beta\mu\nu}\{\frac{1}{2}g^{\alpha p}(\nabla^{\mu}\nabla^{\nu}h^{\beta}_p+\nabla^{\nu}\nabla^{\beta}h^{\mu}_p- -\nabla^{\nu}\nabla_p h^{\beta\mu}-\nabla^{\nu}\nabla^{\mu}h^{\beta}_p-\nabla^{\mu}\nabla^{\beta}h^{\nu}_p+\nabla^{\mu}\nabla_p h^{\beta\nu})\}+R^{\alpha\beta\mu\nu}(\nabla_\alpha\nabla_\beta h_{\mu\nu}+\nabla_\alpha\nabla_\mu h_{\beta\nu}-\nabla_\alpha\nabla_\nu h_{\beta\mu}- \nabla_\beta\nabla_\alpha h_{\mu\nu}--\nabla_\beta\nabla_m uh_{\alpha\nu}+\nabla_\beta\nabla_\nu h_{\alpha\mu})]. \quad (2.30)$$

The special class of higher curvature theories, called the Lovelock gravity are the most general second order gravity theories in higher dimensional space-time. In fact, the higher dimensional space-times are of extreme interest in several candidate frameworks for unifying gravity with other interactions and furthermore, with the Lovelock theories, one can explore the effect of higher curvature terms in black hole thermodynamics without having any concern with complications that arise in true higher derivative theories. Here, we attempt to evolve such a Lovelock theory of higher dimensional dynamical manifold EM under the R.F. mechanism. We, first rewrite the Lovelock Lagrangian (1.6) in more concise form as below:

$$\pounds=\frac{1}{k^2}\sum_{0<n<\frac{D}{2}}\frac{1}{2^n}\prod_{r=1}^{2n}\delta^{\alpha_r}_{\beta_r}R^{\beta_r}_{\alpha_r}\equiv\sum_{0<n<\frac{D}{2}}c_nL^{(n)}. \quad (2.31)$$

Taking the R.F. on both side of equation (2.31), and simplifying under the suitable R.F. techniques described previously, we have

$$\frac{\partial\pounds}{\partial t}\equiv\frac{\partial}{\partial t}\left(\sum_{0<n<\frac{D}{2}}c_nL^{(n)}\right)=\frac{1}{k^2}\sum_{0<n<\frac{D}{2}}\frac{1}{2^n}\prod_{r=1}^{2n}[g^{\alpha_r\mu_r}(g_{\mu_r\beta_r}\frac{\partial}{\partial t}R^{\beta_r}_{\alpha_r}+R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g_{\mu_r\beta_r})+g_{\mu_r\beta_r}\times R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g^{\alpha_r\mu_r}]=\frac{1}{k^2}\sum_{0<n<\frac{D}{2}}\frac{1}{2^n}\prod_{r=1}^{2n}[g^{\alpha_r\mu_r}(g_{\mu_r\beta_r}\frac{g^{p_rq_r}}{2}\{\nabla_{q_r}\nabla_{\alpha_r}h^{\beta_r}_{p_r}+\nabla_{q_r}\nabla^{\beta_r}h_{\alpha_rp_r}-\nabla_{q_r}\nabla_{p_r}h^{\beta_r}_{\alpha_r}- -\nabla_{\alpha_r}\nabla^{\beta_r}h_{q_rp_r}\}+R^{\beta_r}_{\alpha_r}h_{\mu_r\beta_r})+g_{\mu_r\beta_r}R^{\beta_r}_{\alpha_r}g^{\alpha_rl_r}h_{l_rm_r}g^{\mu_rm_r}]. \quad (2.32)$$

From the standpoint of present study, it is evident that the Lovelock tensor (1.9) is nothing but the composition of Ricci and curvature scalar tensors recognized by equations (1.11) and (1.12) respectively. Thus to evolve a non-linear Lovelock heat diffusion equation, we first develop the non-linear heat diffusion equations for the identities (1.11) and (1.12) as follows:
Applying R.F. mechanism for identity (1.11), we have

$$\frac{\partial}{\partial t}R^{(n)}_{\alpha\beta}=\frac{\partial}{\partial t}\left[\frac{n}{2^n}\prod_{r=1}^{2n}g_{\alpha\beta}\delta^{\alpha_r}_{\beta_r}R^{\beta_r}_{\alpha_r}\right]=\frac{n}{2^n}\prod_{r=1}^{2n}[g_{\alpha\beta}\{g^{\alpha_r\mu_r}(g_{\mu_r\beta_r}\frac{\partial}{\partial t}R^{\beta_r}_{\alpha_r}+R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g_{\mu_r\beta_r})+ +g_{\mu_r\beta_r}R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g^{\alpha_r\mu_r}\}+\delta^{\alpha_r}_{\beta_r}R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g_{\alpha\beta}]=\frac{n}{2^n}\prod_{r=1}^{2n}[g_{\alpha\beta}\{g^{\alpha_r\mu_r}g_{\mu_r\beta_r}\frac{g^{p_rq_r}}{2}(\nabla_{q_r}\nabla_{\alpha_r}h^{\beta_r}_{p_r}+\nabla_{q_r}\nabla^{\beta_r}h_{\alpha_rp_r}- -\nabla_{q_r}\nabla_{p_r}h^{\beta_r}_{\alpha_r}-\nabla_{\alpha_r}\nabla^{\beta_r}h_{q_rp_r})+g^{\alpha_r\mu_r}R^{\beta_r}_{\alpha_r}h_{\mu_r\beta_r}+g_{\mu_r\beta_r}R^{\beta_r}_{\alpha_r}g^{\alpha_rl_r}h_{l_rm_r}g^{\mu_rm_r}\}+\delta^{\alpha_r}_{\beta_r}R^{\beta_r}_{\alpha_r}h_{\alpha\beta}]. \quad (2.33)$$

Likewise, applying appropriate R.F. evolution techniques to equation (1.12), so that we could have

$$\frac{\partial}{\partial t}R^{(n)} = \frac{1}{2^n}\prod_{r=1}^{2n}\frac{\partial}{\partial t}\left(\delta^{\alpha_r}_{\beta_r}R^{\beta_r}_{\alpha_r}\right) = \frac{1}{2^n}\prod_{r=1}^{2n}[g^{\alpha_r\mu_r}(g_{\mu_r\beta_r}\frac{\partial}{\partial t}R^{\beta_r}_{\alpha_r} + R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g_{\mu_r\beta_r}) + g_{\mu_r\beta_r}R^{\beta_r}_{\alpha_r}\frac{\partial}{\partial t}g^{\alpha_r\mu_r}]$$

$$= \frac{1}{2^n}\prod_{r=1}^{2n}[g^{\alpha_r\mu_r}g_{\mu_r\beta_r}\frac{g^{p_rq_r}}{2}(\nabla_{q_r}\nabla_{\alpha_r}h^{\beta_r}_{p_r} + \nabla_{q_r}\nabla^{\beta_r}h_{\alpha_rp_r} - \nabla_{q_r}\nabla_{p_r}h^{\beta_r}_{\alpha_r} - \nabla_{\alpha_r}\nabla^{\beta_r}h_{q_rp_r})+$$

$$+ g^{\alpha_r\mu_r}R^{\beta_r}_{\alpha_r}h_{\mu_r\beta_r} + g_{\mu_r\beta_r}R^{beta_r}_{\alpha_r}g^{\alpha_rl_r}h_{l_rm_r}g^{\mu_rm_r}]. \quad (2.34)$$

Eventually, in viwe of flow expressions (2.33), (2.34) and (1.10), we have the following Lovelock heat diffusion equation;

$$\frac{\partial}{\partial t}\mathbb{G}_{\alpha\beta} = \sum_{0<n<\frac{D}{s}}c_n\frac{\partial}{\partial t}G^{(n)}_{\alpha\beta} = \sum_{0<n<\frac{D}{2}}c_n\frac{n}{2^n}\prod_{r=1}^{2n}[g_{\alpha\beta}\{g^{\alpha_r\mu_r}g_{\mu_r\beta_r}\frac{g^{p_rq_r}}{2}(\nabla_{q_r}\nabla_{\alpha_r}h^{\beta_r}_{p_r}+$$

$$\nabla_{q_r}\nabla^{\beta_r}h_{\alpha_rp_r} - -\nabla_{q_r}\nabla_{p_r}h^{\beta_r}_{\alpha_r} - \nabla_{\alpha_r}\nabla^{\beta_r}h_{q_rp_r}) + g^{\alpha_r\mu_r}R^{\beta_r}_{\alpha_r}h_{\mu_r\beta_r} + g_{\mu_r\beta_r}R^{\beta_r}_{\alpha_r}g^{\alpha_rl_r}h_{l_rm_r}g^{\mu_rm_r}\}$$

$$+\delta^{\alpha_r}_{\beta_r}R^{\beta_r}_{\alpha_r}h_{\alpha\beta}]-- \sum_{0<n<\frac{D}{2}}c_n\frac{1}{2^{n+1}}g_{\alpha\beta}\prod_{r=1}^{2n}[g^{\alpha_r\mu_r}g_{\mu_r\beta_r}\frac{g^{p_rq_r}}{2}(\nabla_{q_r}\nabla_{\alpha_r}h^{\beta_r}_{p_r}+\nabla_{q_r}\nabla^{\beta_r}h_{\alpha_rp_r}-\nabla_{q_r}\nabla_{p_r}h^{\beta_r}_{\alpha_r}-$$

$$\nabla_{\alpha_r}\nabla^{\beta_r}h_{q_rp_r}) + +g^{\alpha_r\mu_r}R^{\beta_r}_{\alpha_r}h_{\mu_r\beta_r} + g_{\mu_r\beta_r}R^{beta_r}_{\alpha_r}g^{\alpha_rl_r}h_{l_rm_r}g^{\mu_rm_r}]. \quad (2.35)$$

Concluding Remarks

Here is the brief discussion over some main outcomes of this article written in favor of evolution of Lovelock tensor as a generalized Einstein's tensor and Lovelock gravity under R.F. mechanism:

a In the section (1) A brief digest on the historical evolution of Lovelock theory including a very surprising concept of dynamical manifold has been carried out.

b In the subsection (1.1), we have studied the gravitational field theory based on most general quadratic Lagrangians. In addition, we have gone through the Lovelock Lagrangian and the Lovelock tensor and their significance in the study of gravity theory of dynamical manifold.

c In subsection (1.2), we have pursued generalized trace techniques of Farhoudi (2009) which are very useful to solve the difficulty in case of inhomogeneous Lovelock tensor.

d Section (2) has been the most crucial part of our research. In this section, we have attempted to describe the dynamical manifold EM, from a new perspective called R.F. mechanism. Various time evolving metrics have been discussed and under these time evolving techniques, we have evolved Einstein's equations following Mach's principle. Besides, by employing the time evolving R.F. mechanism, a non-linear heat diffusion equation for Einstein-Hilbert action has been established. Further, an action involving Einstein's gravity coupled with conformally invariant gravity has also been evolved under R.F. techniques. Afterward, a R.F. expression for the Lovelock Lagrangian has been setup. Finally, the Ricci tensor $R^{(n)}_{\alpha\beta}$ and the curvature scalar $R^{(n)}$ have been treated under appropriate R.F. mechanism so that their diffusion equation could be employed to evolve the diffusion equation of Lovelock tensor as a diffusion equation of generalized Einstein tensor.

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