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Weak RIP and Its Application to Compressed Sensing

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Abstract

The first purpose of this paper is to give a sufficient condition under which A obeys the weak RIP and to evaluate the solution of CS using this result. The second is to show that when an $m \times n$ random matrix A satisfies the isotropy property: $E\left(A_{\{k\}}A^{\star}_{\{k\}}\right) = I$ for every row vector $A_{\{k\}}$ of A, $\tilde{A} \equiv \frac{A}{\sqrt{m}}$ always obeys the weak RIP with high probability and it is applicable to the CS theory.

Keywords: Compressed sensing, Isotropy property, Restricted isometry constants, Restricted isometry property, Sparse approximation, Sparse signal recovery, Weak restricted isometry property.

1 Introduction

This paper introduces the theory of compressed sensing (CS). CS theory asserts that one can recover certain signals and images from only a few samples or measurements. Here, we consider

$$\boldsymbol{y} = A\boldsymbol{x}, \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{1.1}$$

where A is an $m \times n$ matrix. Our goal is to reconstruct $\boldsymbol{x} \in \boldsymbol{R}^n$ with good accuracy. We are interested in the ill-posed problem when $m < n$. It is known that when $\boldsymbol{x}$ is sparse, or approximately sparse, and A obeys the restricted isometry property (RIP), one can accurately reconstruct $\boldsymbol{x}$ from the measurements $\boldsymbol{y} = A\boldsymbol{x}$. In fact, the solution $\boldsymbol{x}^{\star}$ to the optimization problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \| \tilde{\boldsymbol{x}} \|_1 \quad \text{subject to} \quad \boldsymbol{y} = A\tilde{\boldsymbol{x}} \tag{1.2}$$

recovers $\boldsymbol{x}$ exactly, where $\| \cdot \|_1$ is the l_1 norm. Furthermore, we extend this method to noisy recovery. Suppose we observe

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{1.3}$$

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where $\boldsymbol{z}$ is an unknown noise term. In this context, we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \| \tilde{\boldsymbol{x}} \|_1 \quad \text{subject to} \quad \| \boldsymbol{y} - A\tilde{\boldsymbol{x}} \|_2 \leq \varepsilon, \tag{1.4}$$

where ε is an upper bound on the size of the noisy contribution and $\| \cdot \|_2$ is the l_2 norm.

Definition 1.1. A matrix A satisfies the RIP of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1-\delta) \| \boldsymbol{a} \|_2^2 \leq \| A\boldsymbol{a} \|_2^2 \leq (1+\delta) \| \boldsymbol{a} \|_2^2 \tag{1.5}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be s-sparse if it has at most s nonzero entries. The minimum of the above constants δ is said to be the isometry constant of A and is denoted by δ_s.

The condition (1.5) is equivalent to requiring that the matrix $A_S^T A_S$ has all of its eigenvalues in $[1-\delta_s, 1+\delta_s]$, where A_S is the $m \times |S|$ matrix composed of these columns for any subset S of $\{1, 2, \cdots, n\}$. Here $|S|$ is number of elements of S. It has been shown that l_1 optimization can recover an unknown signal in noiseless case and noisy case under various sufficient conditions on δ_s or δ_{2s}. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered [[1]]. Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ [2]. Others, $\delta_{2s} < 0.4652$ is used by [3], $\delta_{2s} < 0.4721$ for cases such that s is a multiple of 4 or s is very large by [4], $\delta_{2s} < 0.4734$ for the case such that s is very large by [3] and $\delta_s < 0.307$ by [4]. In a recent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ for the special case such that $n \leq 4s$ [5]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_s < 0.333$ for general case [6]. T. Cai and A. Zhang have improved the sufficient condition to δ_k in case of $k \leq \frac{4}{3}s$, in particular $\delta_{2s} < 0.707$ [7]. Recently, H. Inoue has defined the k-restrictly norm constant $r_k(A)$ of a matrix A and by the rescaling technique, he has proved the sufficient conditions for the rescaled matrix $\tilde{A} \equiv \frac{A}{r_k(A)}$ that if $\tilde{\delta}_s < 0.5$ and $\tilde{\delta}_{2s} < 0.828$, then an unknown compressive signal with noise can be recovered [8].

This paper shows that it is possible to apply CS theory to various fields. For example, when we apply CS to a statistical model, we define A as a basis function matrix and $\boldsymbol{x}$ as a coefficient vector. We have to estimate the coefficient vector and assess this model. In this case, if A is a random matrix, we can not interpret the estimated model. Thus, in order to interpret models, it is important to discuss the method of using a matrix according to the structure of the data and the assessment of estimators. However, the RIP requires a bounded condition number for all submatrices built by selecting s arbitrary columns and the spectral norm of a matrix is generally difficult to calculate. Therefore, it seems useful to weaken the condition of RIP. In [9], E.J. Candès and Y. Plan have introduced the notion of weak RIP which is a generalization of RIP as follows:

Definition 1.2. (Weak RIP) Let $T_0 \subset \{1, 2, \cdots, n\}$ with $|T_0| = s$ and $1 < r < s$. A obeys the weak RIP with respect to T_0 of order r if there exists $0 < \delta < 1$ such that for any subset $R \subset T_0^c$ with $|R| \leq r$,

$$(1-\delta) \| \boldsymbol{x}_{T_0 \cup R} \|_2^2 \leq \| A\boldsymbol{x}_{T_0 \cup R} \|_2^2 \leq (1+\delta) \| \boldsymbol{x}_{T_0 \cup R} \|_2^2 \tag{1.6}$$

for all $\boldsymbol{x} \in \boldsymbol{R}^n$. The minimum of such constants δ is denoted by $\delta_{T_0,r}$.

Roughly speaking the notion of the weak RIP, we choose a suitable location T_0 with $|T_0| = s$ in the columns of the matrix A. We remark that A obeys the RIP of order r, but it does not necessarily obey the RIP of order $(s+r)$. Furthermore, the matrix $A_{T_0 \cup R}$ obeys the inequality (1.6) for any subset R of T_0^c with $|R| = r$. In [9], E.J. Candès and Y. Plan have proved that under the assumptions of isotropy property and incoherence property a random matrix obeys the weak RIP with high probability $1-5e^{-\beta}$

if $m \geq C \log n$ (where C is a constant which only depends on $\beta,\ \delta,\ s,\ r$ and the coherent parameter μ), and have evaluated stochastically the solution of LASSO [10] using the weak RIP and the other properties (the existence of inexact dual vector, the noise correlation bound and etc.) In a recent paper [11], H. Inoue has focused on this notion and evaluate the solution of CS under the assumption of only the weak RIP without the probability, and obtain almost the same results as for the case of the RIP. Thus it seems that the notion of weak RIP is useful in case that we have some information about the data, that is, we have a good location T_0, and it seems better to analyze data using the weak RIP because it is much easier to construct matrices obeying the weak RIP than matrices obeying the RIP. In this paper, we give a sufficient condition under which A obeys the weak RIP with respect to T_0 of order r and evaluate the solution of CS by using a correlative relationship $\theta_{T_0,r}$ of the locations T_0 and T_0^c defined in (2.6). Furthermore, we apply this result to the case of a random matrix satisfying the isotropy property.

2 The Weak RIP and CS

Throughout this paper, let T_0 be a subset of $\{1, 2, \cdots, n\}$ with $|T_0| = s$ and r be a natural number with $0 < r < s$. In this section, we define the coefficient of correlation $\theta_{T_0,r}$ of A_{T_0} and $A_{T_0^c}$ and give a sufficient condition of $\theta_{T_0,r}$ under which A obeys the weak RIP with respect to T_0 of order r and evaluate the solution of CS. We assume the following (i) and (ii):

(i) The submatrix A_{T_0} is nearly isometric, that is, there exists a constant δ $(0 \leq \delta < 1)$ such that

$$(1-\delta)\|\boldsymbol{x}\|_2^2 \leq \|A_{T_0}\boldsymbol{x}\|_2^2 \leq (1+\delta)\|\boldsymbol{x}\|_2^2 \tag{2.1}$$

for each $\boldsymbol{x} \in \boldsymbol{R}^n$ with $\mathrm{supp}\ \boldsymbol{x} \subset T_0$. The minimum of such constants δ is denoted by $\delta(T_0)$.
The matrix A_{T_0} is nearly isometric if and only if it obeys the RIP of order s. It is easily shown that

$$0 < \|A_{T_0}^\star A_{T_0}\| < 2, \quad \|\left(A_{T_0}^\star A_{T_0}\right)^{-1}\| > 1 \tag{2.2}$$

and

$$\delta(T_0) = \begin{cases} 1 - \frac{1}{\|\left(A_{T_0}^\star A_{T_0}\right)^{-1}\|} & \text{if} \quad \|A_{T_0}^\star A_{T_0}\| \leq 1 \\ \max\left(\|A_{T_0}^\star A_{T_0}\| - 1,\ 1 - \frac{1}{\|\left(A_{T_0}^\star A_{T_0}\right)^{-1}\|}\right) = \max(\lambda_1 - 1,\ 1 - \lambda_s) & \text{if} \quad \|A_{T_0}^\star A_{T_0}\| > 1, \end{cases} \tag{2.3}$$

where λ_1 and λ_s are the maximum eigenvalue and the minimum eigenvalue of the positive matrix $A_{T_0}^\star A_{T_0}$, respectively.

(ii) $A_{T_0^c}$ obeys the RIP of order r. Let $\delta_r(T_0^c)$ denote the restricted isometry constant of $A_{T_0^c}$.

We consider the correlative relationship of the submatrices A_{T_0} and $A_{T_0^c}$. Let T be any location of T_0^c with $|T| = r$. Then we define the coefficient of correlation $\theta_{T_0,r}$ of A_{T_0} and A_T by

$$\begin{aligned} \mu\left(T_0,\ T\right) &= \sup\left\{|<A\boldsymbol{x},\ A\boldsymbol{y}>|;\ \ \mathrm{supp}\ \boldsymbol{x} \subset T_0, \mathrm{supp}\ \boldsymbol{y} \subset T, \|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2 = 1\right\}, \\ \mu_{T_0,r} &= \max\left\{\mu(T_0, T);\ \ T \subset T_0^c \ \text{ with } \ |T| = r\right\}. \end{aligned} \tag{2.4}$$

Then we have

$$\mu_{T_0,r} = \max\left\{\|A_T^\star A_{T_0}\|;\ \ T \subset T_0^c \text{ with } |T| = r\right\}. \tag{2.5}$$

Here we define the coefficient of correlation $\theta_{T_0,r}$ as follows:

$$\theta_{T_0,r} = \max\left(\delta(T_0),\ \delta_r(T_0^c),\ \mu_{T_0,r}\right). \tag{2.6}$$

It is easily shown that whenever $r' \leq r$

$$\delta_{r'}(T_0^c) \leq \delta_r(T_0^c) \ \text{ and } \ \mu_{T_0,r'} \leq \mu_{T_0,r}, \tag{2.7}$$

so that

$$\theta_{T_0,r'} \leq \theta_{T_0,r} \tag{2.8}$$

Suppose that A obeys the weak RIP with respect to T_0 of order r. Then it is clear that A_{T_0} is nearly isometric with $\delta(T_0) \leq \delta_{T_0,r}$ and $A_{T_0^c}$ obeys the RIP of order r with $\delta_r(T_0^c) \leq \delta_{T_0,r}$. Furthermore, since

$$| < A\boldsymbol{x}, A\boldsymbol{y} > | \leq \delta_{T_0,r}\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 \tag{2.9}$$

for each $\boldsymbol{x},\ \boldsymbol{y} \in \boldsymbol{R}^n$ with $\mathrm{supp}\ \boldsymbol{x} \subset T_0$ and $\mathrm{supp}\ \boldsymbol{y} \subset T_0^c$ with $|\mathrm{supp}\ \boldsymbol{y}| \leq r$, it follows that $\mu_{T_0,r} \leq \delta_{T_0,r}$. Hence we have

$$\theta_{T_0,r} \leq \delta_{T_0,r}. \tag{2.10}$$

Conversely we have the following

Theorem 2.1. Suppose that A_{T_0} is nearly isometric, $A_{T_0^c}$ obeys the RIP of order r and $\theta_{T_0,r} < \frac{1}{2}$. Then A obeys the weak RIP with respect to T_0 of order r and $\theta_{T_0,r} \leq \delta_{T_0,r} \leq 2\theta_{T_0,r}$.

Proof. Take arbitrary $\boldsymbol{x},\ \boldsymbol{y} \in \boldsymbol{R}^n$ such that $\mathrm{supp}\ \boldsymbol{x} \subset T_0$, $\mathrm{supp}\ \boldsymbol{y} \subset T_0^c$ and $|\mathrm{supp}\ \boldsymbol{y}| = r$. Then, since

$$\begin{aligned} \|A(\boldsymbol{x}+\boldsymbol{y})\|_2^2 &= \|A\boldsymbol{x}\|_2^2 + 2\left\langle A\boldsymbol{x}, A\boldsymbol{y}\right\rangle + \|A\boldsymbol{y}\|_2^2 \\ &\leq (1+\delta(T_0))\|\boldsymbol{x}\|_2^2 + 2\mu_{T_0,r}\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 + (1+\delta_r(T_0^c))\|\boldsymbol{y}\|_2^2 \\ &\leq (1+2\theta_{T_0,r})\left(\|\boldsymbol{x}\|_2^2 + \|\boldsymbol{y}\|_2^2\right) \\ &= (1+2\theta_{T_0,r})\|\boldsymbol{x}+\boldsymbol{y}\|_2^2 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \|A(\boldsymbol{x}+\boldsymbol{y})\|_2^2 &\geq \|A\boldsymbol{x}\|_2^2 - 2| < A\boldsymbol{x}, A\boldsymbol{y} > | + \|A\boldsymbol{y}\|_2^2 \\ &\geq (1-\delta(T_0))\|\boldsymbol{x}\|_2^2 - 2\mu_{T_0,r}\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 + (1-\delta_r(T_0))\|\boldsymbol{y}\|_2^2 \\ &\geq (1-2\theta_{T_0,r})\|\boldsymbol{x}+\boldsymbol{y}\|_2^2, \end{aligned} \tag{2.12}$$

it follows that

$$(1-2\theta_{T_0,r})\|\boldsymbol{x}+\boldsymbol{y}\|_2^2 \leq \|A(\boldsymbol{x}+\boldsymbol{y})\|_2^2 \leq (1+2\theta_{T_0,r})\|\boldsymbol{x}+\boldsymbol{y}\|_2^2, \tag{2.13}$$

which implies Theorem 2.1.

We have the following result for an evaluation of the solution of CS.

Theorem 2.2. Suppose A_{T_0} is nearly isometric, $A_{T_0^c}$ obeys the RIP of order r and

$$2\theta_{T_0,\frac{r}{5}} + \sqrt{\frac{5s}{2r}}\theta_{T_0,r} < 1. \tag{2.14}$$

Then A obeys the weak RIP with respect to T_0 of order r and

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \leq C_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + C_1\varepsilon,$$

where

$$\begin{aligned} C_0 &= \sqrt{\frac{5}{r}}\left(\frac{1-2\theta_{T_0,\frac{r}{5}}+\sqrt{2}\theta_{T_0,r}}{1-2\theta_{T_0,\frac{r}{5}}-\sqrt{\frac{5s}{2r}}\theta_{T_0,r}}\right), \\ C_1 &= \frac{2\left(1+\sqrt{\frac{5s}{2r}}\right)\sqrt{1+\theta_{T_0,\frac{r}{5}}}}{1-2\theta_{T_0,\frac{r}{5}}-\sqrt{\frac{5s}{2r}}\theta_{T_0,r}}. \end{aligned} \tag{2.15}$$

In particular, if $\boldsymbol{x}$ is a T_0-sparse, that is, $\mathrm{supp}\ \boldsymbol{x} \subset T_0$, then

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \le C_1\varepsilon.$$

Proof. We put $\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}$. Then we have

$$\|\boldsymbol{h}\|_2 \le 2\varepsilon \tag{2.16}$$

and by definition of CS optimization

$$\|\boldsymbol{h}_{T_0^c}\|_1 \le 2\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_0}\|_1. \tag{2.17}$$

We consider the following decomposition of $\{1,2,\cdots,n\}$: Let T_1 be the location of the r' largest coefficients of $\boldsymbol{h}_{T_0^c}$, T_2 the location of the $r'' \equiv r - r'$ largest coefficients of $\boldsymbol{h}_{(T_0\cup T_1)^c}$ and T_3 the location of the r'' largest coefficients of $\boldsymbol{h}_{(T_0\cup T_1\cup T_2)^c}$. Repeating this method, $\{1,2,\cdots,n\} = T_0 \cup T_1 \cup \cdots \cup T_{l-1}$, $|T_l| \le r''$. Then, since

$$|h_k^{T_{j-1}}| \ge \max_{k\in T_j}|h_k^{T_j}|,\ \ 2\le j\le l,\ \ 1\le k\le r'',$$

it follows from Proposition 2.1. in [4] that

$$\|\boldsymbol{h}_{T_j}\|_2 \le \frac{1}{\sqrt{r''}}\|\boldsymbol{h}_{T_j}\|_1 + \frac{\sqrt{r''}}{4}\left(|h_1^{T_j}| - |h_{r''}^{T_j}|\right), \quad j\ge 2$$

and

$$\begin{aligned} \sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_2 &\le \frac{1}{\sqrt{r''}}\sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_1 + \frac{\sqrt{r''}}{4}|h_1^{T_2}| \\ &\le \frac{1}{\sqrt{r''}}\|\boldsymbol{h}_{(T_0\cup T_1)^c}\|_1 + \frac{\sqrt{r''}}{4r'}\|\boldsymbol{h}_{T_1}\|_1 \\ &= \frac{1}{\sqrt{r''}}\|\boldsymbol{h}_{T_0^c}\|_1 - \left(\frac{1}{\sqrt{r''}} - \frac{\sqrt{r''}}{4r'}\right)\|\boldsymbol{h}_{T_1}\|_1. \end{aligned} \tag{2.18}$$

By taking $r' = \frac{1}{5}r$ and $r'' = \frac{4}{5}r$, we can obtain the decompsition $\{T_1, T_2, \cdots, T_l\}$ of T_0^c, which is better than those of [4] and [12]. Then it follows from (2.17) and (2.18) that

$$\begin{aligned} \sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_2 &\le \frac{1}{\sqrt{\frac{4}{5}r}}\|\boldsymbol{h}_{T_0^c}\|_1 \\ &\le \sqrt{\frac{5}{r}}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + \sqrt{\frac{5}{4r}}\|\boldsymbol{h}_{T_0}\|_1 \\ &\le \sqrt{\frac{5}{r}}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + \sqrt{\frac{5s}{4r}}\|\boldsymbol{h}_{T_0\cup T_1}\|_2. \end{aligned} \tag{2.19}$$

Since $A_{T_0^c}$ obeys the RIP of order r, it follows that

$$| < A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_j} > | \le \delta_r(T_0^c)\|\boldsymbol{h}_{T_1}\|_2\|\boldsymbol{h}_{T_j}\|_2, \quad j\ge 2. \tag{2.20}$$

By the assumption (2.14), we have $\theta_{T_0,r} < \sqrt{\frac{2r}{5s}} < \frac{1}{2}$. Hence it follows from Theorem 2.1 that A obeys the weak RIP with respect to T_0 of order r. Furthermore, we can show similarly to (2.11) and (2.12) that

$$\left(1 - 2\theta_{T_0,\frac{r}{5}}\right) \|\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \le \|A\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \le \left(1 + 2\theta_{T_0,\frac{r}{5}}\right) \|\boldsymbol{h}_{T_0\cup T_1}\|_2^2, \tag{2.21}$$

which implies by (2.16), (2.7) and (2.20) that

$$\begin{aligned}
&\left(1 - 2\theta_{T_0,\frac{r}{5}}\right) \|\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \\
\le\ & \|A\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \\
\le\ & < A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h} > + \sum_{j\ge 2} | < A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j} > | \\
\le\ & 2\varepsilon\sqrt{1 + 2\theta_{T_0,\frac{r}{5}}}\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \sum_{j\ge 2} | < A\boldsymbol{h}_{T_0}, A\boldsymbol{h}_{T_j} > | + \sum_{j\ge 2} | < A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_j} > | \\
\le\ & 2\varepsilon\sqrt{1 + 2\theta_{T_0,\frac{r}{5}}}\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \mu_{T_0,\frac{4}{5}r}\|\boldsymbol{h}_{T_0}\|_2 \left(\sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_2\right) + \delta_r(T_0^c)\|\boldsymbol{h}_{T_1}\|_2 \left(\sum_{j\ge 2}\boldsymbol{h}_{T_j}\|_2\right) \\
\le\ & 2\varepsilon\sqrt{1 + 2\theta_{T_0,\frac{r}{5}}}\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \sqrt{2}\theta_{T_0,r}\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \left(\sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_2\right),
\end{aligned} \tag{2.22}$$

and by (2.19)

$$\left(1 - 2\theta_{T_0,\frac{r}{5}} - \sqrt{\frac{5s}{2r}}\theta_{T_0,r}\right) \|\boldsymbol{h}_{T_0\cup T_1}\|_2 \le 2\sqrt{1 + \theta_{T_0,\frac{r}{5}}}\varepsilon + \sqrt{\frac{10}{r}}\theta_{T_0,r}\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1. \tag{2.23}$$

Hence we have by (2.14) and (2.23)

$$\begin{aligned}
\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 &\le \|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \|\boldsymbol{h}_{(T_0\cup T_1)^c}\|_2 \\
&\le \|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_2 \\
&\le \left(1 + \sqrt{\frac{5s}{4r}}\right)\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \sqrt{\frac{5}{r}}\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\
&\le C_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + C_1\varepsilon.
\end{aligned}$$

This completes the proof of Theorem 2.2.

We next consider a special case that A satisfies a restricted norm condition:

$$\|A_T\| \le 1, \tag{2.24}$$

for each $T \subset T_0^c$ with $|T| \le r$. Then we have the following

Theorem 2.3. Suppose that $A_{T_0}^\star A_{T_0}$ is invertible, $A_T^\star A_T$ is invertible for every $T \subset T_0^c$ with $|T| \le r$ and

$$2\theta_{T_0,\frac{r}{5}} + \sqrt{\frac{5s}{2r}}\max\left(\theta_{T_0,\frac{4}{5}r},\ \frac{1}{2}\theta_{T_0,r}\right) < 1. \tag{2.25}$$

Then we have

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \le D_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + D_1\varepsilon,$$

where

$$D_0 = \sqrt{\frac{5}{r}}\left(\frac{1-2\theta_{T_0,\frac{r}{5}}+\sqrt{2}\max\left(\theta_{T_0,\frac{4}{5}r},\frac{1}{2}\theta_{T_0,r}\right)}{1-2\theta_{T_0,\frac{r}{5}}-\sqrt{\frac{5s}{2r}}\max\left(\theta_{T_0,\frac{4}{5}r},\frac{1}{2}\theta_{T_0,r}\right)}\right),$$

$$D_1 = \frac{2\sqrt{2}\left(1+\sqrt{\frac{5s}{4r}}\right)}{1-2\theta_{T_0,\frac{r}{5}}-\sqrt{\frac{5s}{2r}}\max\left(\theta_{T_0,\frac{4}{5}r},\frac{1}{2}\theta_{T_0,r}\right)}. \tag{2.26}$$

Proof. Since $\|A_{T_0}\| \leq 1$ and $A^{\star}_{T_0}A_{T_0}$ is invertible, it follows that A_{T_0} is nearly isometric and $\delta(T_0) = 1 - \frac{1}{\|\left(A^{\star}_{T_0}A_{T_0}\right)^{-1}\|}$. Since $\|A_T\| \leq 1$ and $A^{\star}_T A_T$ is invertible for every $T \subset T_0^c$ with $|T| \leq r$, it follows that $A_{T_0^c}$ obeys the RIP of order r and

$$\delta_r(T_0^c) = 1 - \frac{1}{\min\{\| \left(A^{\star}_T A_T\right)^{-1} \|;\ T \subset T_0^c \text{ and } |T| \leq r\}}. \tag{2.27}$$

Since $\|A_{T_0}\| \leq 1$ and $\|A_T\| \leq 1$ for each $T \subset T_0^c$ with $|T| \leq r$, it follows that

$$| < A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_j} > | \leq \frac{1}{2}\delta_r(T_0^c)\|\boldsymbol{h}_{T_1}\|_2\|\boldsymbol{h}_{T_j}\|_2 \quad , j \geq 2 \tag{2.28}$$

and similarly to (2.22)

$$\begin{aligned}
&\left(1-2\theta_{T_0,\frac{r}{5}}\right)\|\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \\
\leq\ & \|A\boldsymbol{h}_{T_0\cup T_1}\|_2^2 \\
\leq\ & 2\sqrt{2}\varepsilon\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \mu_{T_0,\frac{4}{5}r}\|\boldsymbol{h}_{T_0}\|_2\left(\sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2\right) + \frac{1}{2}\delta_r(T_0^c)\|\boldsymbol{h}_{T_1}\|_2\left(\sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2\right) \\
\leq\ & 2\sqrt{2}\varepsilon\|\boldsymbol{h}_{T_0\cup T_1}\|_2 + \sqrt{2}\max\left(\theta_{T_0,\frac{4}{5}r},\frac{1}{2}\theta_{T_0,r}\right)\|\boldsymbol{h}_{T_0\cup T_1}\|_2\left(\sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2\right),
\end{aligned}$$

which implies by (2.19)

$$\begin{aligned}
&\left(1-2\theta_{T_0,\frac{r}{5}}-\sqrt{\frac{5s}{2r}}\max\left(\theta_{T_0,\frac{4}{5}r},\frac{1}{2}\theta_{T_0,r}\right)\right)\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \\
\leq\ & 2\sqrt{2}\varepsilon + \sqrt{\frac{10}{r}}\max\left(\theta_{T_0,\frac{4}{5}r},\frac{1}{2}\theta_{T_0,r}\right)\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1.
\end{aligned}$$

Hence we have

$$\|\boldsymbol{x}^{\star} - \boldsymbol{x}\|_2 \leq D_0\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + D_1\varepsilon.$$

Remark. In case that A does not satisfy the restricted norm condition (2.24), we may obtain a similar result to Theorem 2.3 using a rescaled matrix $\tilde{A} \equiv \frac{A}{\sigma_r(T_0^c)}$, where

$$\sigma_r(T_0^c) = \sup\{\|A_T\|;\ \ T \subset T_0^c \text{ and } |T| = r\}.$$

Such rescaling technique has been used by many authors. In particular, H. Inoue has shown that it is useful to apply the above rescaled matrix $\tilde{A}$ to the theory of CS [8].

3 The Weak RIP and CS with Probability

In this section, we next evaluate the solution of CS in case that a random $m \times n$ matrix A satisfies the isotropy property:

$$E\left(A_{\{k\}}A_{\{k\}}^{\star}\right) = I \tag{3.1}$$

for every row vector $A_{\{k\}}$ of A. We put $|a_{ij}| \le \rho(A)$ (simply, ρ), $1 \le i \le m$, $1 \le j \le n$. Then we have the following

Theorem 3.1. For any $0 < \delta < \frac{1}{2}$, $\tilde{A} \equiv \frac{A}{\sqrt{m}}$ obeys the weak RIP with respect to T_0 order r and $\tilde{\theta}_{T_0,r} \le \delta$ with probability at least $1 - e^{-\beta}$ provided with

$$m \ge \max\left\{\begin{array}{c} \frac{2\left(1+\frac{\delta}{3}\right)}{\delta^2}(\rho s-1)(\beta+\log\ 2s), \\ \frac{2\left(1+\frac{\delta}{3}\right)}{\delta^2}(\rho r-1)\left(\beta + r\log\ \frac{n-s}{r} + r + \log\ 2r\right), \\ \frac{8\rho sr}{\delta^2}\left(\beta+\log\left(n-s\right)+\frac{1}{4}\right) \end{array}\right\}. \tag{3.2}$$

We consider the following equality (3.3) instead of (1.3):

$$\tilde{\boldsymbol{y}} = \tilde{A}\boldsymbol{x} + \tilde{\boldsymbol{z}}, \tag{3.3}$$

where $\tilde{\boldsymbol{y}} \equiv \frac{\boldsymbol{y}}{\sqrt{m}}$, $\tilde{A} \equiv \frac{A}{\sqrt{m}}$ and $\tilde{\boldsymbol{z}} \equiv \frac{\boldsymbol{z}}{\sqrt{m}}$. Since $\{\boldsymbol{a} \in \boldsymbol{R}^n;\ \|\boldsymbol{y} - A\boldsymbol{a}\|_2 \le \varepsilon\} = \{\boldsymbol{a} \in \boldsymbol{R}^n;\ \|\tilde{\boldsymbol{y}} - \tilde{A}\boldsymbol{a}\|_2 \le \frac{\varepsilon}{\sqrt{m}}\}$, it follows that the solution $\boldsymbol{x}^\star$ to the optimization problem is the same as that to (1.4).

Theorem 3.2. The solution $\boldsymbol{x}^\star$ to (1.4) obeys

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \le C_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + C_1\frac{\varepsilon}{\sqrt{m}},$$

with probability with $1 - e^{-\beta}$ provided with

$$m \ge \max\left\{\begin{array}{c} \frac{3\left(2+\sqrt{\frac{5s}{2r}}\right)^2}{2\left(5+\sqrt{\frac{5s}{2r}}\right)}(\rho s-1)(\beta+\log\ 2s), \\ \frac{3\left(2+\sqrt{\frac{5s}{2r}}\right)^2}{2\left(5+\sqrt{\frac{5s}{2r}}\right)}(\rho r-1)\left(\beta + r\log\ \frac{n-s}{r} + r + \log\ 2r\right), \\ 8\left(2+\sqrt{\frac{5s}{2r}}\right)^2\rho sr\left(\beta+\log\ n+\frac{1}{4}\right) \end{array}\right\}. \tag{3.4}$$

We prepare some lemmas to prove Theorem 3.1 and Theorem 3.2. We assume that a random $m \times n$ matrix A satisfies the isotropy property and put $\tilde{A} = \frac{A}{\sqrt{m}}$. Let δ be any real number with $0 < \delta < 1$.

Lemma 3.1. Let T be any subset of $\{1, 2, \cdots, n\}$ with $|T| = k$. Then $\tilde{A}_T$ is nearly isometric for δ with probability $1 - 2k\exp\left\{-\frac{m}{(\rho k-1)}\frac{\delta^2}{2\left(1+\frac{\delta}{3}\right)}\right\}$.

Proof. This is due to Lemma 2.1 in [9]. We simply give the proof. It is shown that

$$\tilde{A}_T^\star\tilde{A}_T - I = \frac{1}{m}\sum_{i=1}^{m}\left(\left(A_{\{i\}T}\right)\left(A_{\{i\}T}\right)^\star - I\right).$$

Putting $X_i = A_{\{i\}T}A_{\{i\}T}^\star - I$, $i = 1, 2, \cdots, m$, we can show that $E(X_i) = 0$, $\|X_i\| \le \rho k - 1$ and $0 \le E(X_i^2) \le (\rho k - 1)I$. Hence it follows from the matrix Bernstein inequality [9] that

$$P\left(\|\tilde{A}_T^\star\tilde{A}_T - I\| \ge \delta\right) \le 2k\exp\left\{-\frac{m}{(\rho k-1)}\frac{\delta^2}{2\left(1+\frac{\delta}{3}\right)}\right\}. \tag{3.5}$$

Lemma 3.2. The following statements (i) and (ii) hold:
(i) $\tilde{A}_{T_0}$ is nearly isometric for δ with probability at least $1-e^{-\beta}$ provided with

$$m \geq \frac{2\left(1+\frac{\delta}{3}\right)}{\delta^2}(\rho s-1)(\beta+\log\ 2s). \tag{3.6}$$

(ii) $\tilde{A}_{T_0^c}$ obeys the RIP of order r and $\delta_r(T_0^c) \leq \delta$ with probability at least $1-e^{-\beta}$ provided with

$$\begin{aligned} m &\geq \frac{2\left(1+\frac{\delta}{3}\right)}{\delta^2}(\rho r-1)(\beta+\log\ {}_{n-s}C_r+\log 2r) \\ &\geq \frac{2\left(1+\frac{\delta}{3}\right)}{\delta^2}(\rho r-1)\left(\beta+r\log\ \frac{n-s}{r}+r+\log 2r\right). \end{aligned} \tag{3.7}$$

Proof. (i) This follows from Lemma 3.1.
(ii) By Lemma 3.1 we have

$$\begin{aligned} & P\left(\bigcup_T\left\{\|\tilde{A}_T^\star\tilde{A}_T-I\|\geq\delta\right\}\right) \\ \leq\ & \sum_T P\left(\|\tilde{A}_T^\star\tilde{A}_T-I\|\geq\delta\right) \\ \leq\ & {}_{n-s}C_r 2r\exp\left(-\frac{m}{(\rho r-1)}\frac{\delta^2}{2\left(1+\frac{\delta}{3}\right)}\right) \\ =\ & \exp\left(-\frac{m}{(\rho r-1)}\frac{\delta^2}{2\left(1+\frac{\delta}{3}\right)}+\log\ \left({}_{n-s}C_r 2r\right)\right) \\ \leq\ & \exp\left(-\frac{m}{(\rho r-1)}\frac{\delta^2}{2\left(1+\frac{\delta}{3}\right)}+\log\ \left(\left(\frac{e(n-s)}{r}\right)^r 2r\right)\right) \\ =\ & \exp\left(-\frac{m}{(\rho r-1)}\frac{\delta^2}{2\left(1+\frac{\delta}{3}\right)}+r\log\ \frac{n-s}{r}+r+\log 2r\right), \end{aligned} \tag{3.8}$$

where T moves all subsets of T_0^c with $|T|=r$, which implies (2).

Lemma 3.3. We have

$$\tilde{\theta}_{T_0,r} \leq \delta \tag{3.9}$$

with probability at least $1-e^{-\beta}$ provided with

$$m \geq \frac{8\rho sr}{\delta^2}\left(\beta+\log(n-s)+\frac{1}{4}\right). \tag{3.10}$$

Proof. By Lemma 2.5 in [9], we have

$$P\left(\max_{i\in T_0^c}\|\tilde{A}_{T_0}^\star \boldsymbol{a}_i\|_2\geq t\right) \leq (n-s)\exp\left(-\frac{mt^2}{8\rho s}+\frac{1}{4}\right). \tag{3.11}$$

Hence we have

$$\begin{aligned} P\left(\max_{i\in T_0^c}\|\tilde{A}_{T_0}^\star \boldsymbol{a}_i\|_2\geq t\right) &\geq P\left(\max_{i\in T}\|\tilde{A}_{T_0}^\star \boldsymbol{a}_i\|_2\geq t\right) \\ &= P\left(\left(\bigcap_{i\in T}\left\{\|\tilde{A}_{T_0}^\star \boldsymbol{a}_i\|_2<t\right\}\right)^c\right) \\ &= 1-P\left(\bigcap_{i\in T}\left\{\tilde{A}_{T_0}^\star \boldsymbol{a}_i\|_2<t\right\}\right), \end{aligned} \tag{3.12}$$

where T moves all subset of T_0^c with $|T| = r$, which implies by (3.11) that

$$\begin{aligned} P\left(\bigcap_{i\in T}\left\{\|\tilde{A}_{T_0}^{\star}\boldsymbol{a}_i\|_2 < t\right\}\right) &\geq 1 - P\left(\max_{i\in T_0^c}\|\tilde{A}_{T_0}^{\star}\boldsymbol{a}_i\|_2 \geq t\right) \\ &\geq 1-(n-s)\exp\left(-\frac{mt^2}{8\rho s}+\frac{1}{4}\right). \end{aligned} \tag{3.13}$$

Since

$$\begin{aligned} \|\tilde{A}_{T_0}^{\star}\tilde{A}_T\boldsymbol{w}\|_2 &= \|\sum_{i\in T} w_i\tilde{A}_{T_0}^{\star}\boldsymbol{a}_i\|_2 \\ &\leq \sum_{i\in T}|w_i|\|\tilde{A}_{T_0}^{\star}\boldsymbol{a}_i\|_2 \\ &\leq \|\boldsymbol{w}\|_2\left(\sum_{i\in T}\|\tilde{A}_{T_0}^{\star}\boldsymbol{a}_i\|_2^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.14}$$

for each $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)^T \in \boldsymbol{R}^n$, it follows from (3.13) that

$$\tilde{\theta}_{T_0,r} = \max\left\{\|\tilde{A}_{T_0}^{\star}\tilde{A}_T\|;\ \ T\subset T_0^c\ \text{ with }\ |T| = r\right\} \leq t\sqrt{r} \tag{3.15}$$

with at least probability $1-(n-s)\exp\left(-\frac{mt^2}{8\rho s}+\frac{1}{4}\right)$. Putting $t = \frac{\delta}{\sqrt{r}}$, Lemma 3.3 holds.

Proof of Theorem 3.1. This follows from Lemma 3.2, Lemma 3.3 and Theorem 2.1.

Proof of Theorem 3.2. We put $\delta = \frac{1}{2+\sqrt{\frac{5s}{2r}}}$. Then Theorem 3.2 follows from Theorem 3.1 and Theorem 2.2.

Remark. In [9], E.J. Candès and Y. Plan have shown that if A satisfies the isotropy property and

$$m \geq C_\delta\beta\rho\max\ \left(s\log\ s\rho, r\log\ n\left(\log\ r\right)^2\log\left(r\rho\log\ n\right)\right), \tag{3.16}$$

then A obeys the weak RIP with respect to T_0 of order r with probability $\left(1-5e^{-\beta}\right)$. In Theorem 3.2, we have obtained by a simple proof that A obeys the weak RIP with respect to T_0 of order r with probability $\left(1-e^{-\beta}\right)$ if m satisfies the inequality (3.2) and the condition (3.2) of m is better than (3.16) if n is sufficient large for s (for example, $n \geq e^{(\log r)^s}$).

4 Conclusions

In this paper, we study the weak RIP and its application to compressed sensing. In Theorem 2.1, we have introduced the property of a correlative relationship $\theta_{T_0,r}$ of the locations T_0 and T_0^c. In Theorem 2.2 and Theorem 2.3, we have given a sufficient condition under which A obeys the weak RIP with respect to T_0 of order r and have evaluated the solution of CS by using a correlative relationship $\theta_{T_0,r}$. In Theorem 3.1 and Theorem 3.2, we have evaluated the solution of CS with probability in case that a random matrix satisfying the isotropy property.

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Competing Interests

The author declares that no competing interests exist.

References

[1] Candès EJ, Tao T. Decoding by linear programming, IEEE Transactions on Information Theory. 2005;51:4203–4215.

[2] Foucart S, Lai M. Sparsest solutions of underdetermined linear systems via l_q-minimization for $0 < q \leq 1$, Applied and Computational Harmonic Analysis. 2009;26:395-407.

[3] Foucart S. A note on guaranteed sparse recovery via l_1-minimization, Applied and Computational Harmonic Analysis. 2010;29:97–103.

[4] Cai T, Wang L, Xu G. New bounds for restricted isometry constants, IEEE Transactions on Information Theory. 2010;56:4388-4394.

[5] Mo Q, Li S. New bounds on the restricted isometry constant δ_{2k}, Applied and Computational Harmonic Analysis. 2011;31:460-468.

[6] Cai T, Zhang A. Sharp RIP Bound for Sparse Signal and Low-Rank Matrix Recovery, Applied and Computational Harmonic Analysis. 2013;35:74-93.

[7] Cai T, Zhang A. Sparse representation of a polytope and recovery of sparse signals and low-rank matrices, Technical Report. 2013.

[8] Inoue H. Sufficient conditions for CS-recovery, British Journal of Mathematics & Computer Science, (In press).

[9] Candès EJ, Plan Y. A probabilistic and RIPless theory of compressed sensing, IEEE Transactions on Information Theory. 2010;57:7235-7254.

[10] Tibshirani R. Regression shrinkage and selection via the Lasso, Journal of the Royal Statistical Society Series B 1996;58:267-288.

[11] Inoue H. A generalization of the restricted isometry property and applications to compressed sensing, Journal of Math-for-Industry. 2013B;5:129-133.

[12] Candès EJ. The restricted isometry property and its implications for compressed sensing, C. R. Acad Sci Ser I. 2008;346:589-592.

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