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Numerical Approaches for Tenth and Twelfth Order Linear and Nonlinear Differential Equations

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Abstract

The aim of this paper is to solve the tenth and twelfth order linear and nonlinear boundary value problems numerically by the Galerkin weighted residual technique with two point boundary conditions. The well known Bernstein polynomials are exploited as basis functions in the technique and thus the basis functions are needed to modify into a new set of basis functions where the *Dirichlet* types of boundary conditions are satisfied. The method is developed as a rigorous matrix formulation. Numerical examples, available in the literature, are considered to implement the proposed technique. The comparison shows that the present method is more efficient and yields better results.

Keywords: Galerkin method, tenth and twelfth order BVP, linear and nonlinear differential equations, bernstein polynomials.

1 Introduction

The existence and uniqueness theorem of solutions of boundary value problems (BVP) was discussed extensively by Agarwal [1] without any numerical examples. In the literature of BVPs

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we observe that the higher order differential equations arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences. Particularly, eighth, tenth and more even higher order BVP arise in hydromagnetic stability analyses which are available in [2]. But few researchers have paid their attentions to solve high order BVP numerically. Finite difference methods for the solution of such problems were developed by Boutayeb and Twizell [3], Djidjeli et al. [4], and Twizell et al. [5] but these solutions were found only at specific grid points while Siddiqi and Iftikhar [6] solved these problems by homotopy analysis method (HAM). Inc and Evans [7] solved eighth order, Siddiqi et al. [8] solved seventh order BVPs using Adomian decomposition method whereas Siddiqi and Twizell [9] developed spline solutions for eighth order problems. Also nonpolynomial spline solution technique was introduced by Siddiqi and Akram [10] for these BVP. Besides these, Siddiqi and Iftikhar [11,12] solved seventh order BVP by the variation of parameters and Adomian decomposition method. From the literature we observe that the tenth and twelfth order BVP has been attempted to solve numerically by a few researchers, namely, Siddiqi and Twizell [13] solved using tenth degree spline while Siddiqi and Akram [14] presented the solutions by eleventh degree spline polynomials. Also variational iteration technique was introduced by Siddiqi et al. [15] for solving these tenth order problems. On the other hand, Siddiqi and Twizell [16] solved the twelfth order BVPs using twelfth degree splines while Siddiqi and Akram [17] developed the solutions of twelfth order BVPs by applying thirteen degree splines. Al- Kudri and Mulhem [18] derived the numerical solutions of twelfth-order BVPs using adomain decomposition method. Mirmoradi et al. [19] solved twelfth-order BVPs by the homotopy perturbation method. Also Noor and Mohy-ud-Din [20] used variational iteration method for solving these BVPs by applying He's polynomials. The modified decomposition method has been used extensively only by Wazwaz [21] to find the solutions of nonlinear BVP of higher order while Iftikhar et al. [22] used differential transformed method to solve thirteen order BVP. Recently, Hossain et al. [23] have studied the eleventh order BVPs using some piecewise polynomials through Galerkin method with high accuracy. Thus, our aim is to solve both the linear and nonlinear BVPs of order tenth and twelfth by a suitable and reliable efficient method.

In the present paper, we apply Galerkin method [24] with Bernstein [25] polynomials as basis functions for the numerical solution of the general tenth and twelfth order linear differential equations:

$$a_{10}\frac{d^{10}u}{dx^{10}}+a_9\frac{d^9u}{dx^9}+a_8\frac{d^8u}{dx^8}+a_7\frac{d^7u}{dx^7}+a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}+a_3\frac{d^3u}{dx^3}$$
$$+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=r\ ,\ a<x<b \tag{1a}$$

subject to the boundary conditions:

$$u(a)=A_0,u(b)=B_0,u'(a)=A_1,u'(b)=B_1,u''(a)=A_2,u''(b)=B_2,$$
$$u'''(a)=A_3,u'''(b)=B_3,u^{(iv)}(a)=A_4,u^{(iv)}(b)=B_4 \tag{1b}$$

and

$$a_{12}\frac{d^{12}u}{dx^{12}}+a_{11}\frac{d^{11}u}{dx^{11}}+a_{10}\frac{d^{10}u}{dx^{10}}+a_9\frac{d^9u}{dx^9}+a_8\frac{d^8u}{dx^8}+a_7\frac{d^7u}{dx^7}+a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}$$
$$+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=r,\ a<x<b \tag{2a}$$

subject to the boundary conditions:

$$u(a)=A_0, u(b)=B_0, u'(a)=A_1, u'(b)=B_1, u''(a)=A_2, u''(b)=B_2, u'''(a)=A_3,$$
$$u'''(b)=B_3, u^{(iv)}(a)=A_4, u^{(iv)}(b)=B_4, u^{(v)}(a)=A_5, u^{(v)}(b)=B_5 \tag{2b}$$

where $A_i, B_i, i=0,1,2,3,4,5$ are finite real constants, $a_i\ (i=0,1,\cdots,12)$ and r are all continuous and differentiable functions defined on the interval [*a, b*]. However, we present a short description on Bernstein polynomials in section 2. We also formulate Galerkin method in matrix form in section 3. Several numerical examples and their results are given in section 4 to verify the proposed method. The numerical solutions, obtained by the present method, are compared with the results of the methods available in the literature. The conclusions are described in section 5.

2 Bernstein Polynomials

The Bernstein polynomials, general form of degree *n* over the finite interval $[a,b]$, is defined by [25]:

$$B_{i,n}(x)=\binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \qquad a\le x\le b \qquad i=0,1,2,\ldots,n.$$

The first 17 Bernstein polynomials over the interval $[0,\ 1]$, which will be used in this paper, are given bellow:

$B_0(x)=(1-x)^{16}$ $\quad B_6(x)=8008(1-x)^{10}x^6$ $\quad B_{12}(x)=1820(1-x)^4x^{12}$

$B_1(x)=16(1-x)^{15}x$ $\quad B_7(x)=11440(1-x)^9x^7$ $\quad B_{13}(x)=560(1-x)^3x^{13}$

$B_2(x)=120(1-x)^{14}x^2$ $\quad B_8(x)=12870(1-x)^8x^8$ $\quad B_{14}(x)=120(1-x)^2x^{14}$

$B_3(x)=560(1-x)^{13}x^3$ $\quad B_9(x)=11440(1-x)^7x^9$ $\quad B_{15}(x)=16(1-x)x^{15}$

$B_4(x)=1820(1-x)^{12}x^4$ $\quad B_{10}(x)=8008(1-x)^6x^{10}$ $\quad B_{16}(x)=x^{16}$

$B_5(x)=4368(1-x)^{11}x^5$ $\quad B_{11}(x)=4368(1-x)^5x^{11}$

The Bernstein polynomials also satisfy the following properties:

$(i)\quad B_{i,n}(x)=0$ if $i<0$ or $i>n$.

$(ii)\quad \sum_{i=0}^{n}B_{i,n}(x)=1$

$(iii)\quad B_{i,n}(a)=B_{i,n}(b)=0,\quad i=1,2,\ldots,n-1$

Thus, we use these polynomials in the trail functions of the Galerkin method, to be described in the following section, since it satisfies the corresponding homogeneous form of the *Dirichlet* boundary conditions.

3 Description of the Method

To solve the boundary value problem (1) by the Galerkin method we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i \, B_{i,n}(x) \tag{3}$$

and the corresponding weighted residual equations are

$$\int_a^b \left[a_{10} \frac{d^{10}\widetilde{u}}{dx^{10}} + a_9 \frac{d^9\widetilde{u}}{dx^9} + a_8 \frac{d^8\widetilde{u}}{dx^8} + a_7 \frac{d^7\widetilde{u}}{dx^7} + a_6 \frac{d^6\widetilde{u}}{dx^6} + a_5 \frac{d^5\widetilde{u}}{dx^5} + a_4 \frac{d^4\widetilde{u}}{dx^4} + a_3 \frac{d^3\widetilde{u}}{dx^3} + a_2 \frac{d^2\widetilde{u}}{dx^2} + a_1 \frac{d\widetilde{u}}{dx} + a_0\widetilde{u} - r \right] B_{j,n}(x)dx = 0 \tag{4}$$

where $\theta_0(x)$ is specified by the essential boundary conditions, $B_{i,n}(x)$ are the Bernstein polynomials which must satisfy the corresponding homogeneous boundary conditions such that $B_{i,n}(a) = B_{i,n}(b) = 0$ for each $i = 1, 2, \ldots n$. Integrating by parts the terms up to second derivative on the left hand side of (4), we obtain

$$\begin{aligned}\int_a^b a_{10} \frac{d^{10}\widetilde{u}}{dx^{10}} B_{j,n}(x)dx = &-\left[\frac{d}{dx}\left[a_{10}B_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_{10}B_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_{10}B_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}\left[a_{10}B_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_a^b - \left[\frac{d^5}{dx^5}\left[a_{10}B_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_a^b + \left[\frac{d^6}{dx^6}\left[a_{10}B_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_a^b \\ &- \left[\frac{d^7}{dx^7}\left[a_{10}B_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_a^b + \left[\frac{d^8}{dx^8}\left[a_{10}B_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^9}{dx^9}\left[a_{10}B_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx\end{aligned} \tag{5}$$

$$\begin{aligned}\int_a^b a_9 \frac{d^9\widetilde{u}}{dx^9} B_{j,n}(x)dx = &-\left[\frac{d}{dx}\left[a_9B_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_9B_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_9B_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}\left[a_9B_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_a^b - \left[\frac{d^5}{dx^5}\left[a_9B_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_a^b + \left[\frac{d^6}{dx^6}\left[a_9B_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_a^b \\ &- \left[\frac{d^7}{dx^7}\left[a_9B_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^8}{dx^8}\left[a_9B_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx\end{aligned} \tag{6}$$

$$\int_a^b a_8 \frac{d^8\tilde{u}}{dx^8} B_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_8 B_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_8 B_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_8 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b$$
$$+\left[\frac{d^4}{dx^4}\left[a_8 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^5}{dx^5}\left[a_8 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^6}{dx^6}\left[a_8 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b$$
$$-\int_a^b \frac{d^7}{dx^7}\left[a_8 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{7}$$

$$\int_a^b a_7 \frac{d^7\tilde{u}}{dx^7} B_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_7 B_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_7 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_7 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b$$
$$+\left[\frac{d^4}{dx^4}\left[a_7 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^5}{dx^5}\left[a_7 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^6}{dx^6}\left[a_7 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{8}$$

$$\int_a^b a_6 \frac{d^6\tilde{u}}{dx^6} B_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_6 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b$$
$$+\left[\frac{d^4}{dx^4}\left[a_6 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^5}{dx^5}\left[a_6 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{9}$$

$$\int_a^b a_5 \frac{d^5\tilde{u}}{dx^5} B_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_5 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_5 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b$$
$$+\int_a^b \frac{d^4}{dx^4}\left[a_5 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{10}$$

$$\int_a^b a_4 \frac{d^4\tilde{u}}{dx^4} B_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_4 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_4 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b -$$
$$\int_a^b \frac{d^3}{dx^3}\left[a_4 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{11}$$

$$\int_a^b a_3 \frac{d^3\tilde{u}}{dx^3} B_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_3 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[a_3 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{12}$$

$$\int_a^b a_2 \frac{d^2\tilde{u}}{dx^2} B_{j,n}(x)dx = -\int_a^b \frac{d}{dx}\left[a_2 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{13}$$

Substituting the eqns. (5) – (13) into eqn. (4) and using approximation for $\tilde{u}(x)$ given in eqn. (3) and after rearranging the terms for the resulting equations we get a system of equations in matrix form

$$\sum_{i=1}^{n} D_{i,j}\, a_i = F_j,\ j = 1, 2, ..., n \tag{14a}$$

where,

$$D_{i,j}=\int_a^b\Bigg\{\Bigg[-\frac{d^9}{dx^9}\left[a_{10}B_{j,n}(x)\right]+\frac{d^8}{dx^8}\left[a_9B_{j,n}(x)\right]-\frac{d^7}{dx^7}\left[a_8B_{j,n}(x)\right]+\frac{d^6}{dx^6}\left[a_7B_{j,n}(x)\right]-\frac{d^5}{dx^5}\left[a_6B_{j,n}(x)\right]+\frac{d^4}{dx^4}\left[a_5B_{j,n}(x)\right]$$
$$-\frac{d^3}{dx^3}\left[a_4B_{j,n}(x)\right]+\frac{d^2}{dx^2}\left[a_3B_{j,n}(x)\right]-\frac{d}{dx}\left[a_2B_{j,n}(x)\right]+a_1B_{j,n}(x)\Bigg]\frac{d}{dx}[B_{i,n}(x)]+a_0B_{i,n}(x)B_{j,n}(x)\Bigg\}dx$$
$$-\left[\frac{d}{dx}\left[a_{10}B_{j,n}(x)\right]\frac{d^8}{dx^8}[B_{i,n}(x)]\right]_{x=b}+\left[\frac{d}{dx}\left[a_{10}B_{j,n}(x)\right]\frac{d^8}{dx^8}[B_{i,n}(x)]\right]_{x=a}+\left[\frac{d^2}{dx^2}\left[a_{10}B_{j,n}(x)\right]\frac{d^7}{dx^7}[B_{i,n}(x)]\right]_{x=b}$$
$$-\left[\frac{d^2}{dx^2}\left[a_{10}B_{j,n}(x)\right]\frac{d^7}{dx^7}[B_{i,n}(x)]\right]_{x=a}-\left[\frac{d^3}{dx^3}\left[a_{10}B_{j,n}(x)\right]\frac{d^6}{dx^6}[B_{i,n}(x)]\right]_{x=b}+\left[\frac{d^3}{dx^3}\left[a_{10}B_{j,n}(x)\right]\frac{d^6}{dx^6}[B_{i,n}(x)]\right]_{x=a}$$
$$+\left[\frac{d^4}{dx^4}\left[a_{10}B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=b}-\left[\frac{d^4}{dx^4}\left[a_{10}B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=a}-\left[\frac{d}{dx}\left[a_9B_{j,n}(x)\right]\frac{d^7}{dx^7}[B_{i,n}(x)]\right]_{x=b}$$
$$+\left[\frac{d}{dx}\left[a_9B_{j,n}(x)\right]\frac{d^7}{dx^7}[B_{i,n}(x)]\right]_{x=a}+\left[\frac{d^2}{dx^2}\left[a_9B_{j,n}(x)\right]\frac{d^6}{dx^6}[B_{i,n}(x)]\right]_{x=b}-\left[\frac{d^2}{dx^2}\left[a_9B_{j,n}(x)\right]\frac{d^6}{dx^6}[B_{i,n}(x)]\right]_{x=a}$$
$$-\left[\frac{d^3}{dx^3}\left[a_9B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=b}+\left[\frac{d^3}{dx^3}\left[a_9B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=a}-\left[\frac{d}{dx}\left[a_8B_{j,n}(x)\right]\frac{d^6}{dx^6}B_{i,n}(x)\right]_{x=b}$$
$$+\left[\frac{d}{dx}\left[a_8B_{j,n}(x)\right]\frac{d^6}{dx^6}[B_{i,n}(x)]\right]_{x=a}+\left[\frac{d^2}{dx^2}\left[a_8B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=b}-\left[\frac{d^2}{dx^2}\left[a_8B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=a}$$
$$-\left[\frac{d}{dx}\left[a_7B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=b}+\left[\frac{d}{dx}\left[a_7B_{j,n}(x)\right]\frac{d^5}{dx^5}[B_{i,n}(x)]\right]_{x=a} \quad (14b)$$

$$F_j=\int_a^b\Bigg\{rB_{j,n}(x)+\Bigg[\frac{d^9}{dx^9}\left[a_{10}B_{j,n}(x)\right]-\frac{d^8}{dx^8}\left[a_9B_{j,n}(x)\right]+\frac{d^7}{dx^7}\left[a_8B_{j,n}(x)\right]-\frac{d^6}{dx^6}\left[a_7B_{j,n}(x)\right]$$
$$+\frac{d^5}{dx^5}\left[a_6B_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[a_5B_{j,n}(x)\right]\Bigg]+\frac{d^3}{dx^3}\left[a_4B_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[a_3B_{j,n}(x)\right]+\frac{d}{dx}\left[a_2B_{j,n}(x)\right]$$
$$-a_1B_{j,n}(x)\Bigg]\frac{d\theta_0}{dx}-a_0\theta_0B_{j,n}(x)\Bigg\}dx+\left[\frac{d}{dx}\left[a_{10}B_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=b}-\left[\frac{d}{dx}\left[a_{10}B_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=a}$$
$$-\left[\frac{d^2}{dx^2}\left[a_{10}B_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b}+\left[\frac{d^2}{dx^2}\left[a_{10}B_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a}+\left[\frac{d^3}{dx^3}\left[a_{10}B_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}$$
$$-\left[\frac{d^3}{dx^3}\left[a_{10}B_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}-\left[\frac{d^4}{dx^4}\left[a_{10}B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}+\left[\frac{d^4}{dx^4}\left[a_{10}B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}$$
$$+\left[\frac{d}{dx}\left[a_9B_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b}-\left[\frac{d}{dx}\left[a_9B_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a}-\left[\frac{d^2}{dx^2}\left[a_9B_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}$$

$$+\left[\frac{d^2}{dx^2}\left[a_9 B_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}+\left[\frac{d^3}{dx^3}\left[a_9 B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}-\left[\frac{d^3}{dx^3}\left[a_9 B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=a}$$

$$+\left[\frac{d}{dx}\left[a_8 B_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}-\left[\frac{d}{dx}\left[a_8 B_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}-\left[\frac{d^2}{dx^2}\left[a_8 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}$$

$$+\left[\frac{d^2}{dx^2}\left[a_8 B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=a}+\left[\frac{d}{dx}\left[a_7(x) B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}-\left[\frac{d}{dx}\left[a_7 B_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=a}$$

$$+\left[\frac{d^5}{dx^5}\left[a_{10} B_{j,n}(x)\right]\right]_{x=b}\times B_4-\left[\frac{d^5}{dx^5}\left[a_{10} B_{j,n}(x)\right]\right]_{x=a}\times A_4-\left[\frac{d^6}{dx^6}\left[a_{10} B_{j,n}(x)\right]\right]_{x=b}\times B_3$$

$$+\left[\frac{d^6}{dx^6}\left[a_{10} B_{j,n}(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^7}{dx^7}\left[a_{10} B_{j,n}(x)\right]\right]_{x=b}\times B_2-\left[\frac{d^7}{dx^7}\left[a_{10} B_{j,n}(x)\right]\right]_{x=a}\times A_2$$

$$-\left[\frac{d^8}{dx^8}\left[a_{10} B_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^8}{dx^8}\left[a_{10} B_{j,n}(x)\right]\right]_{x=b}\times A_1-\left[\frac{d^4}{dx^4}\left[a_9 B_{j,n}(x)\right]\right]_{x=b}\times B_4$$

$$+\left[\frac{d^4}{dx^4}\left[a_9 B_{j,n}(x)\right]\right]_{x=a}\times A_4+\left[\frac{d^5}{dx^5}\left[a_9 B_{j,n}(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^5}{dx^5}\left[a_9 B_{j,n}(x)\right]\right]_{x=a}\times A_3$$

$$-\left[\frac{d^6}{dx^6}\left[a_9 B_{j,n}(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^6}{dx^6}\left[a_9 B_{j,n}(x)\right]\right]_{x=a}\times A_2+\left[\frac{d^7}{dx^7}\left[a_9 B_{j,n}(x)\right]\right]_{x=b}\times B_1$$

$$-\left[\frac{d^7}{dx^7}\left[a_9 B_{j,n}(x)\right]\right]_{x=a}\times A_1+\left[\frac{d^3}{dx^3}\left[a_8 B_{j,n}(x)\right]\right]_{x=b}\times B_4-\left[\frac{d^3}{dx^3}\left[a_8 N_{j,n}(x)\right]\right]_{x=a}\times A_4$$

$$-\left[\frac{d^4}{dx^4}\left[a_8 B_{j,n}(x)\right]\right]_{x=b}\times B_3+\left[\frac{d^4}{dx^4}\left[a_8 B_{j,n}(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^5}{dx^5}\left[a_8 B_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$-\left[\frac{d^5}{dx^5}\left[a_8 B_{j,n}(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^6}{dx^6}\left[a_8 B_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^6}{dx^6}\left[a_8 B_{j,n}(x)\right]\right]_{x=b}\times A_1$$

$$-\left[\frac{d^2}{dx^2}\left[a_7 B_{j,n}(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^2}{dx^2}\left[a_7 B_{j,n}(x)\right]\right]_{x=a}\times A_4+\left[\frac{d^3}{dx^3}\left[a_7 B_{j,n}(x)\right]\right]_{x=b}\times B_3$$

$$-\left[\frac{d^3}{dx^3}\left[a_7 B_{j,n}(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^4}{dx^4}\left[a_7 B_{j,n}(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^4}{dx^4}\left[a_7 B_{j,n}(x)\right]\right]_{x=a}\times A_2$$

$$+\left[\frac{d^5}{dx^5}\left[a_7 B_{j,n}(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^5}{dx^5}\left[a_7 B_{j,n}(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\right]_{x=b}\times B_4$$

$$-\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\right]_{x=a}\times A_4-\left[\frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\right]_{x=b}\times B_3+\left[\frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\right]_{x=a}\times A_3$$

$$+\left[\frac{d^3}{dx^3}\left[a_6 B_{j,n}(x)\right]\right]_{x=b} \times B_2 - \left[\frac{d^3}{dx^3}\left[a_6 B_{j,n}(x)\right]\right]_{x=a} \times A_2 - \left[\frac{d^4}{dx^4}\left[a_6 B_{j,n}(x)\right]\right]_{x=b} \times B_1$$
$$+\left[\frac{d^4}{dx^4}\left[a_6 B_{j,n}(x)\right]\right]_{x=a} \times A_1 + \left[\frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\right]_{x=b} \times B_3 - \left[\frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\right]_{x=a} \times A_3$$
$$-\left[\frac{d^2}{dx^2}\left[a_5 B_{j,n}(x)\right]\right]_{x=b} \times B_2 + \left[\frac{d^2}{dx^2}\left[a_5 B_{j,n}(x)\right]\right]_{x=a} \times A_2 + \left[\frac{d^3}{dx^3}\left[a_5 B_{j,n}(x)\right]\right]_{x=b} \times B_1$$
$$-\left[\frac{d^3}{dx^3}\left[a_5 B_{j,n}(x)\right]\right]_{x=a} \times A_1 + \left[\frac{d}{dx}\left[a_4 B_{j,n}(x)\right]\right]_{x=b} \times B_2 - \left[\frac{d}{dx}\left[a_4 B_{j,n}(x)\right]\right]_{x=a} \times A_2$$
$$-\left[\frac{d^2}{dx^2}\left[a_4 B_{j,n}(x)\right]\right]_{x=b} \times B_1 + \left[\frac{d^2}{dx^2}\left[a_4 B_{j,n}(x)\right]\right]_{x=b} \times A_1 + \left[\frac{d}{dx}\left[a_3 B_{j,n}(x)\right]\right]_{x=b} \times B_1$$
$$-\left[\frac{d}{dx}\left[a_3 B_{j,n}(x)\right]\right]_{x=a} \times A_1 \qquad (14c)$$

Solving the system (14a), we obtain the values of a_i which are then used into (3) to get the approximate solution of the BVP (1). In the same way, we can construct a system for twelfth order BVP stated in eqn. (2a) with the boundary conditions described in eqn. (2b).

In the case of nonlinear BVP, we first calculate the initial values from the system (14) on neglecting the nonlinear terms and then Newton's iterative method is exploited for the next approximation. This procedure is described via the numerical examples in the following section 4.

4 Test Examples

To verify our proposed method we consider some linear and nonlinear BVPs consisting of both tenth and twelfth order differential equations. All the calculations, in this section, are performed by ***MATLAB* 10**. Let $\tilde{u}_n(x)$ be the approximate solution of *n* polynomials and let $\delta < 10^{-13}$, then the convergence of linear BVP is calculated as

$$E = \left|\tilde{u}_{n+1}(x) - \tilde{u}_n(x)\right| < \delta$$

The convergence of nonlinear BVP is given by

$$\left|\tilde{u}_n^{N+1} - \tilde{u}_n^{N}\right| < \delta$$

where δ is less than 10^{-12} and N is the Newton's iteration number.

Example 1: We consider the linear BVP of tenth order [14]:

$$\left.\begin{array}{l}\dfrac{d^{10}u}{dx^{10}} - (x^2 - 2x)u = 10\cos x - (x-1)^3 \sin x, \quad -1 \le x \le 1 \\ u(-1) = 2\sin 1,\ u(1) = 0,\ u'(-1) = -2\cos 1 - \sin 1, u'(1) = \sin 1, \\ u''(-1) = 2\cos 1 - 2\sin 1, u''(1) = 2\cos 1,\ u'''(-1) = 2\cos 1 + 3\sin 1, u'''(1) = -3\sin 1, \\ u^{(iv)}(-1) = -4\cos 1 + 2\sin 1, u^{(iv)}(1) = -4\cos 1 \end{array}\right\} \quad (15)$$

The analytical solution of this BVP is $u(x) = (x-1)\sin x$. The maximum absolute errors, using different number of polynomials, by the present method and the previous results obtained so far, are summarized in Table 1.

Table 1. Maximum absolute errors of example 1

Our method using Bernstein polynomials		**Siddiqi and Akram [14]**
No. of polynomials (n)		
12	8.647×10^{-10}	
13	7.961×10^{-13}	2.13×10^{-8}
14	5.998×10^{-14}	
15	7.772×10^{-16}	

Now the exact and approximate solutions are depicted in Fig. 1 of example 1 for $n = 15$.

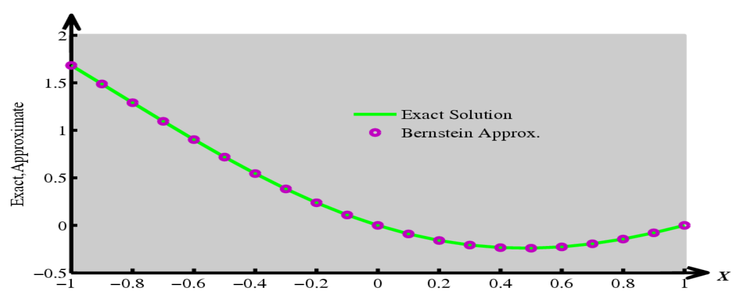

Fig. 1. Graphical representation of exact and approximate solutions of example 1

Example 2: Consider the linear BVP of tenth order [13,14,15]:

$$\left.\begin{array}{l}\dfrac{d^{10}u}{dx^{10}} + u = -10(2x\sin x - 9\cos x), \quad -1 \le x \le 1 \\ u(-1) = u(1) = 0,\ u'(-1) = -2\cos 1, u'(1) = 2\cos 1, u''(-1) = 2\cos 1 - 4\sin 1, \\ u''(1) = 2\cos 1 - 4\sin 1, u'''(-1) = 6\cos 1 + 6\sin 1, u'''(1) = -6\cos 1 - 6\sin 1, \\ u^{(iv)}(-1) = -12\cos 1 + 8\sin 1 = u^{(iv)}(1) \end{array}\right\} \quad (16)$$

The analytical solution of the BVP is, $u(x) = (x^2 - 1)\cos x$. Using the method outlined in section 3, the maximum absolute errors and the existing results obtained in [13,14,15], are shown in Table 2.

Table 2. Maximum absolute errors of example 2

Our method using Bernstein polynomials		References results
No. of polynomials (*n*)	**Results**	
12	9.341×10^{-9}	2.65×10^{-4} ; Siddiqi and Twizell [13]
13	8.034×10^{-12}	8.85×10^{-8} ; Siddiqi and Akram [14]
14	8.999×10^{-13}	4.24×10^{-7} ; Siddiqi et al. [15]
15	9.992×10^{-16}	

Example 3: Consider the linear differential equation of twelfth order [16,17]:

$$\frac{d^{12}u}{dx^{12}} + xu = -(120 + 23x + x^3)e^x, \quad 0 \le x \le 1 \tag{17a}$$

subject to the boundary conditions

$$\begin{aligned} &u(0) = u(1) = 0,\ u'(0) = 1, u'(1) = -e, u''(0) = 0, u''(1) = -4e, u'''(0) = -3, u'''(1) = -9e, \\ &u^{(iv)}(0) = -8, u^{(iv)}(1) = -16e, u^{(v)}(0) = -15, u^{(v)}(1) = -25 \end{aligned} \tag{17b}$$

The analytic solution of the above problem is, $u(x) = x(1-x)e^x$. The maximum absolute errors by the present method are summarized in Table 3. We depict the exact and approximate solutions in Fig. 2 of example 3 for $n = 17$.

Table 3. Maximum absolute errors of example 3

Present method using Bernstein		References results
No. of polynomials (*n*)	**Results**	
14	8.532×10^{-14}	5.582×10^{-3} ; Siddiqi and Twizell [16]
15	8.327×10^{-16}	7.38×10^{-9} ; Siddiqi and Akram [17]
16	7.216×10^{-16}	
17	4.163×10^{-17}	

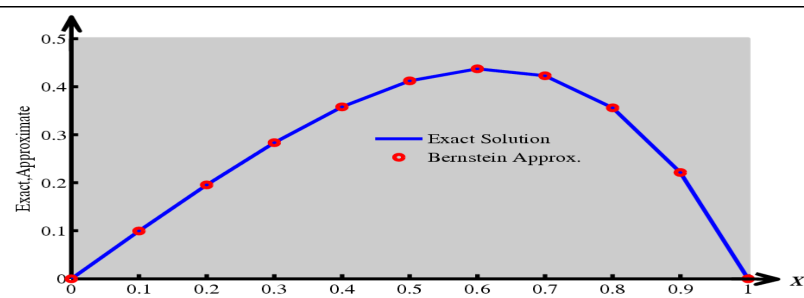

Fig. 2. Graphical representation of exact and approximate solutions of example 3

Example 4: Consider the tenth order nonlinear differential equation [21]

$$\frac{d^{10}u}{dx^{10}} = u^2 e^{-x},\ 0 \le x \le 1 \tag{18}$$

subject to boundary conditions

$$\left.\begin{array}{l} u(0)=1,\ u(1)=e,\ u'(0)=1, u'(1)=e, u''(0)=1, u''(1)=e, u'''(0)=1, u'''(1)=e \\ u^{(iv)}(0)=1, u^{(iv)}(1)=e, u^{(v)}(0)=1, u^{(v)}(1)=e \end{array}\right\} \quad (19)$$

The exact solution of this BVP is, $u(x)=e^{x}$. To solve the differential equation (18) numerically we approximate the solution of $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n} a_i\, B_{i,n}(x),\ \ n\geq 1 \qquad (20)$$

Here $\theta_0(x)=1-x(1-e)$ is specified by the essential boundary conditions in (19). Also $B_{i,n}(0)=B_{i,n}(1)=0$ for each $i=1,2,\ldots,n$. Using (20) into equation (18), the Galerkin weighted residual equations are

$$\int_0^1\left[\frac{d^{10}\tilde{u}}{dx^{10}}-\tilde{u}^2 e^{-x}\right]B_{k,n}(x)dx=0, k=1,2,\cdots,n \qquad (21)$$

Integrating first term of (21) by parts, we obtain

$$\begin{aligned}\int_0^1 \frac{d^{10}\tilde{u}}{dx^{10}}B_{k,n}(x)\,\mathrm{dx} = &-\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 \\ &+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5B_{k,n}(x)}{dx^5}\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6B_{k,n}(x)}{dx^6}\frac{d^3\tilde{u}}{dx^3}\right]_0^1 \\ &-\left[\frac{d^7B_{k,n}(x)}{dx^7}\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^8B_{k,n}(x)}{dx^8}\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^9\,B_{k,n}(x)}{dx^9}\frac{d\tilde{u}}{dx}dx\end{aligned} \qquad (22)$$

Putting (22) into equation (21) and using approximation given in eqn. (20), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\Bigg[\int_0^1\Bigg[&-\frac{d^9B_{k,n}(x)}{dx^9}\frac{dB_{i,n}(x)}{dx}-2\theta_0\,e^{-x}\,B_{i,n}(x)B_{k,n}(x)-\sum_{j=1}^{n}a_j(B_{i,n}(x)B_{j,n}(x)B_{k,n}(x))e^{-x}\Bigg]dx-\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8B_{i,n}(x)}{dx^8}\right]_{x=1} \\ &+\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8B_{i,n}(x)}{dx^8}\right]_{x=0}+\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7B_{i,n}(x)}{dx^7}\right]_{x=1}-\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7B_{i,n}(x)}{dx^7}\right]_{x=0}-\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6B_{i,n}(x)}{dx^6}\right]_{x=1} \\ &+\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6B_{i,n}(x)}{dx^6}\right]_{x=0}+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5B_{i,n}(x)}{dx^5}\right]_{x=1}+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5B_{i,n}(x)}{dx^5}\right]_{x=0}\Bigg]a_i\end{aligned}$$

$$=\int_0^1\left[\frac{d^9B_{k,n}(x)}{dx^9}\frac{d\theta_0}{dx}+\theta_0{}^2e^{-x}B_{k,n}(x)\right]dx+\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=1}-\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=0}-\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$
$$+\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^5B_{k,n}(x)}{dx^5}\right]_{x=1}\times e-\left[\frac{d^5B_{k,n}(x)}{dx^5}\right]_{x=0}-\left[\frac{d^6B_{k,n}(x)}{dx^6}\right]_{x=1}\times e+\left[\frac{d^6B_{k,n}(x)}{dx^6}\right]_{x=0}$$
$$+\left[\frac{d^7B_{k,n}(x)}{dx^7}\right]_{x=1}\times e-\left[\frac{d^7B_{k,n}(x)}{dx^7}\right]_{x=0}-\left[\frac{d^8B_{k,n}(x)}{dx^8}\right]_{x=1}\times e+\left[\frac{d^8B_{k,n}(x)}{dx^8}\right]_{x=0} \qquad (23)$$

The above equation (23) is equivalent to the matrix form

$$(D+B)A=G \qquad (24a)$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k, respectively, given by

$$d_{i,k}=\int_0^1\left[-\frac{d^9B_{k,n}(x)}{dx^9}\frac{dB_{i,n}(x)}{dx}-2\theta_0e^{-x}\,B_{i,n}(x)B_{k,n}(x)\right]dx-\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8B_{i,n}(x)}{dx^8}\right]_{x=1}$$
$$+\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8B_{i,n}(x)}{dx^8}\right]_{x=0}+\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7B_{i,n}(x)}{dx^7}\right]_{x=1}-\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7B_{i,n}(x)}{dx^7}\right]_{x=0}$$
$$-\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6B_{i,n}(x)}{dx^6}\right]_{x=1}+\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6B_{i,n}(x)}{dx^6}\right]_{x=0}+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5B_{i,n}(x)}{dx^5}\right]_{x=1}$$
$$+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5B_{i,n}(x)}{dx^5}\right]_{x=0} \qquad (24b)$$

$$b_{i,k}=-\sum_{j=1}^{n}a_j\int_0^1(B_{i,n}(x)B_{j,n}(x)B_{k,n}(x))\,e^{-x}\,dx \qquad (24c)$$

$$g_k=\int_0^1\left[\frac{d^9B_{k,n}(x)}{dx^9}\frac{d\theta_0}{dx}+\theta_0{}^2e^{-x}B_{k,n}(x)\right]dx+\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=1}-\left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=0}-\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$
$$+\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d^3B_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$+\left[\frac{d^4B_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^5B_{k,n}(x)}{dx^5}\right]_{x=1}\times e-\left[\frac{d^5B_{k,n}(x)}{dx^5}\right]_{x=0}-\left[\frac{d^6B_{k,n}(x)}{dx^6}\right]_{x=1}\times e+\left[\frac{d^6B_{k,n}(x)}{dx^6}\right]_{x=0}$$
$$+\left[\frac{d^7B_{k,n}(x)}{dx^7}\right]_{x=1}\times e-\left[\frac{d^7B_{k,n}(x)}{dx^7}\right]_{x=0}-\left[\frac{d^8B_{k,n}(x)}{dx^8}\right]_{x=1}\times e+\left[\frac{d^8B_{k,n}(x)}{dx^8}\right]_{x=0} \qquad (24d)$$

The initial values of the parameters a_i are calculated by neglecting the nonlinear term in (18). For the initial coefficients we solve the system

$$DA=G \tag{25a}$$

where,

$$d_{i,k} = \int_0^1 -\frac{d^9 B_{k,n}(x)}{dx^9}\frac{dN_{i,n}(x)}{dx}dx - \left[\frac{dB_{k,n}(x)}{dx}\frac{d^8 B_{i,n}(x)}{dx^8}\right]_{x=1} + \left[\frac{dB_{k,n}(x)}{dx}\frac{d^8 B_{i,n}(x)}{dx^8}\right]_{x=0}$$
$$+\left[\frac{d^2 B_{k,n}(x)}{dx^2}\frac{d^7 B_{i,n}(x)}{dx^7}\right]_{x=1} - \left[\frac{d^2 B_{k,n}(x)}{dx^2}\frac{d^7 B_{i,n}(x)}{dx^7}\right]_{x=0} - \left[\frac{d^3 B_{k,n}(x)}{dx^3}\frac{d^6 B_{i,n}(x)}{dx^6}\right]_{x=1}$$
$$+\left[\frac{d^3 B_{k,n}(x)}{dx^3}\frac{d^6 B_{i,n}(x)}{dx^6}\right]_{x=0} + \left[\frac{d^4 B_{k,n}(x)}{dx^4}\frac{d^5 B_{i,n}(x)}{dx^5}\right]_{x=1} + \left[\frac{d^4 B_{k,n}(x)}{dx^4}\frac{d^5 B_{i,n}(x)}{dx^5}\right]_{x=0} \tag{25b}$$

$$g_k = \int_0^1 \frac{d^9 B_{k,n}(x)}{dx^9}\frac{d\theta_0}{dx}dx + \left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=1} - \left[\frac{dB_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=0} - \left[\frac{d^2 B_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$
$$+\left[\frac{d^2 B_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=0} + \left[\frac{d^3 B_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=1} - \left[\frac{d^3 B_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=0} - \left[\frac{d^4 B_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$+\left[\frac{d^4 B_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=0} + \left[\frac{d^5 B_{k,n}(x)}{dx^5}\right]_{x=1}\times e - \left[\frac{d^5 B_{k,n}(x)}{dx^5}\right]_{x=0} - \left[\frac{d^6 B_{k,n}(x)}{dx^6}\right]_{x=1}\times e + \left[\frac{d^6 B_{k,n}(x)}{dx^6}\right]_{x=0}$$
$$+\left[\frac{d^7 B_{k,n}(x)}{dx^7}\right]_{x=1}\times e - \left[\frac{d^7 B_{k,n}(x)}{dx^7}\right]_{x=0} - \left[\frac{d^8 B_{k,n}(x)}{dx^8}\right]_{x=1}\times e + \left[\frac{d^8 B_{k,n}(x)}{dx^8}\right]_{x=0} \tag{25c}$$

Now, the initial values of a_i are computed from eqn. (25a), then used eqn. (24a) to obtain new estimation a_i. Continuing this iteration process until the desired values of the parameters are achieved. Now substitute these new values into eqn. (20) to obtain an approximate solution of the BVP (18). Using 12 Bernstein with six Newton's iterations, we summarized the values at different points of the domain of the problem in Table 4 to compare with the existing modified decomposition method [21]. In this case, our method is better than modified decomposition method [21]. The exact and approximate solutions are depicted in Fig. 3 of example 4 for $n=12$.

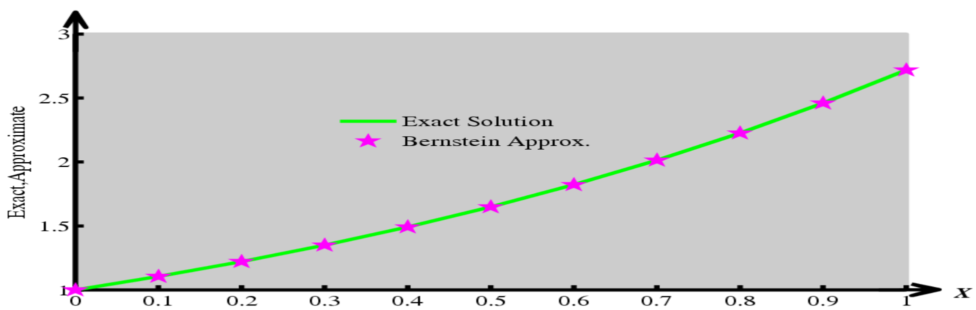

Fig. 3. Graphical representation of exact and approximate solutions of example 4

Table 4. Numerical results of example 4 using 6 iterations

x	Exact	12 Bernstein polynomials	Modified decomposition method [21]
0.0	1.0000000000	0.000000E-000	0.00000
0.1	1.1051709181	5.436494E-012	1.41E-06
0.2	1.2214027582	7.342582E-013	2.69E-06
0.3	1.3498588076	9.542484E-012	3.70E-06
0.4	1.4918246976	1.738262E-012	4.35E-06
0.5	1.6487212707	4.990510E-012	4.58E-06
0.6	1.8221188004	2.407930E-012	4.36E-06
0.7	2.0137527075	4.307570E-013	3.71E-06
0.8	2.2255409285	7.753470E-012	2.69E-06
0.9	2.4596031112	3.203970E-012	1.42E-06
1.0	2.7182818285	0.000000E-000	2.00E-09

Example 5: Consider the **nonlinear** BVP of twelfth order [21]:

$$\frac{d^{12}u}{dx^{12}} = 2e^{x}u^{2} + \frac{d^{3}u}{dx^{3}},\ 0 \le x \le 1 \tag{24a}$$

$$u(0)=1,\ u(1)=e^{-1}, u''(0)=1, u''(1)=e^{-1}, u^{(iv)}(0)=1, u^{(iv)}(1)=e^{-1}, u^{(vi)}(0)=1, u^{(vi)}(1)=e^{-1}$$
$$u^{(viii)}(0)=1, u^{(viii)}(1)=e^{-1}, u^{(x)}(0)=1, u^{(x)}(1)=e^{-1} \tag{24b}$$

The exact solution of this BVP is $u(x)=e^{-x}$. Following the proposed method illustrated in section 3 as well as in example 4; the maximum absolute errors for this problem are summarized in Table 5.

Table 5. Numerical results for example 5 using 6 iterations

x	Exact	13, Bernstein polynomials	Modified decomposition method [21]
0.0	1.0000000000	0.000000E-000	0.00000
0.1	1.1051709181	5.436494E-012	-1.41E-06
0.2	1.2214027582	7.342582E-013	-2.69E-06
0.3	1.3498588076	9.542484E-012	-3.70E-06
0.4	1.4918246976	1.738262E-012	-4.35E-06
0.5	1.6487212707	4.990510E-012	-4.58E-06
0.6	1.8221188004	2.407930E-012	-4.36E-06
0.7	2.0137527075	4.307570E-013	-3.71E-06
0.8	2.2255409285	7.753470E-012	-2.69E-06
0.9	2.4596031112	3.203970E-012	2.00E-09
1.0	2.7182818285	0.000000E-000	

The exact and approximate solutions are illustrated in Fig. 4 of example 5 for $n = 13$.

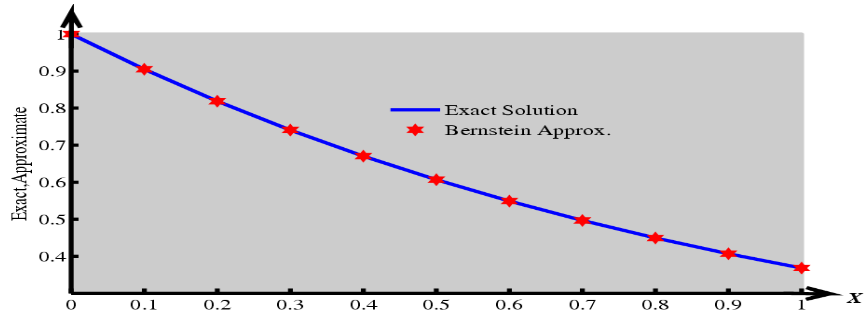

Fig. 4. Graphical representation of exact and approximate solutions of example 5

5 Conclusion

In this paper, Galerkin method has been applied for the numerical solution of tenth and twelfth-order linear and nonlinear BVPs with Bernstein polynomials as basis functions. The numerical examples which are available in the literature have been considered for comparison. From the tables and graphical representation, we observe that our proposed method provides better results than the earlier results obtained by the various researchers. We may conclude that the approximate and analytical solutions are coinciding when the few Bernstein polynomials have been used in the trial function. The algorithm can be coded easily and may be applied to solve any higher order BVP.

Competing Interests

Authors have declared that no competing interests exist.

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